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Rates of Convergence in a Central Limit Theorem for Stochastic Processes Defined by Differential Equations with a Small Parameter

M. A. KOURITZIN

Universität Freiburg

AND

A. J. HEUNIS

University of Waterloo

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Consider the random ordinary differential equation in $\Re^d$

$$\dot{X}^\varepsilon(\tau) = F(X^\varepsilon(\tau), \tau/\varepsilon) \qquad \text{subject to} \quad X^\varepsilon(0) = x_0, \tag{1}$$

where $\varepsilon > 0$ and $\{F(x, t, \omega), t \geqslant 0\}$ is a stochastic process indexed by x in $\Re^d$ which is regular to ensure that there is a unique solution $X^\varepsilon(\cdot, \omega)$ on the interval $0 \leqslant \tau \leqslant 1$ for almost all ω. In a classical paper Khas'minskii (*Theory Probab. Appl.* **11** (1966), 211–228) shows, under broad regularity conditions covering many physical problems of interest, that one can associate with the above equation a certain non-random "averaged" ordinary differential equation

$$\dot{x}^0(\tau) = \bar{F}(x^0(\tau)) \quad \text{subject to} \quad x^0(0) = x_0 \tag{2}$$

such that (i) $\lim_{\varepsilon \to 0} \sup_{0 \leqslant \tau \leqslant 1} E[|X^\varepsilon(\tau) - x^0(\tau)|] = 0$ and (ii) if $Y^\varepsilon(\tau) \triangleq \varepsilon^{-1/2}(X^\varepsilon(\tau) - x^0(\tau))$, then the family of processes $\{Y^\varepsilon(\tau), 0 \leqslant \tau \leqslant 1\}$ converges weakly to a certain limiting Gauss–Markov process $\{\hat{Y}^0(\tau), 0 \leqslant \tau \leqslant 1\}$ as $\varepsilon \to 0$. In this paper we establish a rate of convergence for the central limit theorem in (ii) under conditions only slightly more restrictive than those required by Khas'minskii; in particular, $\{F(x, t, \omega), t \geqslant 0\}$ is allowed to be strong mixing and non-stationary. The rate of convergence is given by a polynomial bound of the form $O(\varepsilon^\lambda)$, for some constant $\lambda > 0$, on the Prohorov distance between the distribution measures (in the space of continuous functions defined for $0 \leqslant \tau \leqslant 1$), generated by the processes $\{Y^\varepsilon(\tau), 0 \leqslant \tau \leqslant 1\}$ and $\{Y^0(\tau), 0 \leqslant \tau \leqslant 1\}$.

1. INTRODUCTION

Consider the following random ordinary differential equation in $\Re^d$

$$\dot{Z}^\varepsilon(t) = \varepsilon F(Z^\varepsilon(t), t) \qquad \text{subject to} \quad Z^\varepsilon(0) = x_0, \tag{1.1}$$

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where $\{F(x, t, \omega),\ t \geqslant 0\}$ is a $\mathfrak{R}^d$-valued "mixing" stochastic process for each x in $\mathfrak{R}^d$ which is regular enough to ensure that, for each $\varepsilon > 0$, there is a unique solution $Z^\varepsilon(\tau, \omega)$ on the interval $0 \leqslant t \leqslant 1/\varepsilon$ for almost all ω. The limiting behaviour (if any) of the solution of (1.1) as $\varepsilon \to 0$ is very relevant to problems in diverse areas of physics and engineering, such as celestial mechanics, theory of nonlinear oscillations, and recursive stochastic algorithms which are much used in problems of control and data communications (for an extensive treatment of the latter see Benveniste *et al.* [1]). The basic intuitive idea is: when the limit

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T EF(x, t)\, dt \triangleq \bar{F}(x) \tag{1.2}$$

exists for each x in $\mathfrak{R}^d$ then it seems reasonable to expect that the function $t \to x^0(\varepsilon t)$ arising from the solution of the non-random ordinary differential equation

$$\dot{x}^0(\tau) = \bar{F}(x^0(\tau)) \qquad \text{subject to} \quad x^0(0) = x_0 \tag{1.3}$$

(assumed for the moment to exist over the interval $0 \leqslant \tau \leqslant 1$ and be unique) approximates, in some appropriate sense, the solution $Z^\varepsilon(\cdot)$ of (1.1) over the inverval $0 \leqslant t \leqslant 1/\varepsilon$ for *small* values of the parameter $\varepsilon > 0$. It is usual to introduce the substitution $X^\varepsilon(\tau) \triangleq Z^\varepsilon(\tau/\varepsilon)$, $0 \leqslant \tau \leqslant 1$, in which case (1.1) takes the form

$$\dot{X}^\varepsilon(\tau) = F(X^\varepsilon(\tau), \tau/\varepsilon) \qquad \text{subject to} \quad X^\varepsilon(0) = x_0, \tag{1.4}$$

and the problem becomes one of comparing the solutions $X^\varepsilon(\cdot, \omega)$ and $x^0(\cdot)$ over the bounded interval $[0, 1]$ as $\varepsilon \to 0$. Khas'minskii [17] shows, under broad regularity conditions covering many physical problems of interest, that (i) $\lim_{\varepsilon \to 0} \sup_{0 \leqslant \tau \leqslant 1} E\,|X^\varepsilon(\tau) - x^0(\tau)| = 0$ and (ii) if $Y^\varepsilon(\tau, \omega) \triangleq \varepsilon^{-1/2}(X^\varepsilon(\tau, \omega) - x^0(\tau))$, then the family of processes $\{Y^\varepsilon(\tau),\ 0 \leqslant \tau \leqslant 1\}$ converges weakly to a certain limiting Gauss–Markov process $\{\hat{Y}^0(\tau)\}$ (whose complete characterisation is given by (3.6) in [17]—see also (2.15) in Section 2 of this note) as $\varepsilon \to 0$. This latter result can be regarded as an analogue of the classical functional central limit theorem of Donsker.

Rates of convergence associated with Donsker's functional central limit theorem have been obtained by Prohorov [21, Chap. 4], Borovkov [4, Theorem 1], Gorodetskii [13, Theorem 1], Yurinskii [24, Section 2], and Borovkov and Sakhanenko [5, Theorem 4], among others. These rates of convergence all assume the form of some polynomial bound on the Prohorov distance between the function space probability measures in $C[0, 1]$ (the space of continuous functions defined over the unit interval), generated by standard Brownian motion and the usual process with continuous "polygonal" sample paths obtained from a normalised running sum of independent random variables. In view of these rates of convergence for Donsker's theorem, it is reasonable to try to establish similar bounds for the functional central limit theorem of Khas'minskii indicated above. The purpose of this note is to show, under conditions only slightly more restrictive than those assumed by Khas'minskii [17], that there is some absolute constant $\lambda > 0$ such that

$$\Pi(\mathscr{L}(Y^{\varepsilon}), \mathscr{L}(\hat{Y}^{0})) = O(\varepsilon^{\lambda}), \tag{1.5}$$

where $\mathscr{L}(Y^{\varepsilon})$ denotes the probability measure generated by the stochastic process $\{Y^{\varepsilon}(\tau), 0 \leqslant \tau \leqslant 1\}$ on the Borel sets of $C[0, 1]$ (and similarly for $\mathscr{L}(\hat{Y}^{0})$) and $\Pi(P, Q)$ is the Prohorov distance between two probability measures P and Q on the Borel sets of $C[0, 1]$.

Besides having intrinsic interest, this bound is also useful in applications. Indeed, (1.5) can be used together with the Strassen theorem on marginals of probability measures (see Dudley [10, Theorem 1]), an approximation theorem of Berkes and Philipp [2, Theorem 1], and a functional law of the iterated logarithm for Gaussian processes due to T. Lai [18, Theorem 1], to relate a.s. the $C[0, 1]$ accumulation points of the set $\{Y^{\varepsilon}(\cdot, \omega)(2 \log \log \varepsilon^{-1})^{-1/2}, \varepsilon > 0\}$ to the unit ball of the reproducing kernel Hilbert space generated by the limiting Gauss–Markov process $\{\hat{Y}^{0}(\tau), 0 \leqslant \tau \leqslant 1\}$. We hope to show this development in a later note.

This note is organised as follows: In Section 2 we state the regularity conditions which will be assumed throughout and compare these with the regularity conditions used by Khas'minskii [17]. The bound (1.5) is proved in Section 3. Following Section 3 are seven appendices where technical results which support the main result in Section 3 are developed. The arrangement of these appendices is as follows: Appendix 1 demonstrates the range of applicability of the conditions in Section 2; Appendix 2 summarizes various facts about the Prohorov metric; Appendix 3 collects an assortment of necesary theorems from contemporary probability theory; Appendix 4 contains moment bounds for strong mixing processes; in Appendix 6 an auxiliary multivariate central limit theorem is developed; and Appendices 5 and 7 are a miscellany of various technical lemmas. Because of the rather large number of supporting results in these appendices we preface the statement of each with a brief indication of the use to which that result is put in the proof of (1.5). The results in the appendices can be referenced at will and are always stated in a manner which is self-contained once the reader is familiar with the basic conditions in Section 2.

2. Conditions

Suppose that $(\Omega, \mathscr{F}, P)$ is a probability space on which is defined a system of $\Re^d$-valued processes $\{F(x, s, \omega),\ s \geqslant 0\}$ indexed by $x \in \Re^d$ and jointly measurable in (s, ω) on $[0, \infty) \times \Omega$ for each x. The following conditions will be assumed throughout this note:

(C0) There exists a P-null set $\Lambda_1 \in \mathscr{F}$ such that for each $\omega \notin \Lambda_1$,

$$\int_0^t |F(0, s, \omega)|\, ds < \infty \qquad \text{for all} \quad 0 \leqslant t < \infty$$

(henceforth, for any vector $X = (X_1 \cdots X_d) \in \Re^d$ we write $|X| \triangleq \max_{i=1 \cdots d} |X_i|$).

(C1) There exists a P-null set Λ_2 and a constant $\bar{N} > 0$ such that $x \to F(x, t)$ is twice continuously differentiable for each $t \geqslant 0$ and $\omega \notin \Lambda_2$ and, moreover,

$$\sup_{\omega \notin \Lambda_2} \sup_{x \in \Re^d} \sup_{t \geqslant 0} \left| \frac{\partial F_i}{\partial x_j}(x, t, \omega) \right| < \bar{N} \tag{2.1}$$

and

$$\sup_{\omega \notin \Lambda_2} \sup_{x \in \Re^d} \sup_{t \geqslant 0} \left| \frac{\partial^2 F_i}{\partial x_j\, \partial x_k}(x, t, \omega) \right| < \bar{N} \tag{2.2}$$

for all integers $1 \leqslant i, j, k \leqslant d$.

In view of condition (C0) and inequality (2.1) the random ordinary differential equation

$$\dot{x}(\tau) = F(x(\tau), \tau/\varepsilon) \qquad \text{subject to} \quad x(0) = x_0, \tag{2.3}$$

has a unique solution $X^\varepsilon(\tau, \omega)$ defined on $0 \leqslant \tau \leqslant 1$ for all $\varepsilon > 0$ and $\omega \notin \Lambda_1 \cup \Lambda_2$ (see, for example, Theorem 3.5 in Chap. II of Reid [22]). The initial condition x_0 is held fixed throughout.

(C2) There exist σ-algebras $\{\mathscr{F}_s^t,\ 0 \leqslant s \leqslant t \leqslant \infty\}$ in Ω such that for each x and $t \geqslant 0$, $F(x, t)$ is $\mathscr{F}_t^t$-measurable with respect to ω, where

(i) $\mathscr{F}_s^t \subset \mathscr{F}$ for all $0 \leqslant s \leqslant t \leqslant \infty$

(ii) $\mathscr{F}_s^t \subset \mathscr{F}_u^v$ for all $0 \leqslant u \leqslant s \leqslant t \leqslant v \leqslant \infty$

(iii) The $\mathscr{F}_s^t$ are strong mixing in the sense of Rosenblatt. That is, if $\alpha(\tau)$ is defined for all $\tau \geqslant 0$ by

$$\alpha(\tau) \triangleq \sup_{t \geqslant 0} \sup_{\xi \eta} |E\xi\eta - E\xi E\eta|$$

where $\sup_{\xi\eta}$ is taken over all real-valued $\mathscr{F}_0^t$-measurable functions ξ and real-valued $\mathscr{F}_{t+\tau}^{\infty}$-measurable functions η such that $|\xi|, |\eta| < 1$) then $\alpha(\tau) \to 0$ as $\tau \to \infty$. Note that $0 \leqslant \alpha(\tau) \leqslant 2$ for all $\tau \geqslant 0$, and $\alpha(\cdot)$ is non-increasing.

(C3) There exists constant $\delta > 0$ such that

$$M \triangleq \sup_{t \geqslant 0} \|F(0, t)\|_{8+4\delta} < \infty,$$

where, for any d-dimensional random vector $X = (X_1 \cdots X_d)$ and $1 \leqslant r < \infty$, we write $\|X\|_r \triangleq E^{1/r}(|X|^r)$.

(C4) There are constants $\theta > 4$ and $\eta > 0$ such that the Rosenblatt mixing coefficient, $\alpha(\cdot)$, defined in (C2) satisfies

$$\alpha(\tau) \leqslant \eta \tau^{-\theta(1+2\delta^{-1})} \qquad \text{for all} \quad \tau \geqslant 1, \tag{2.4}$$

where δ is the constant of (C3).

Note that δ in (C3) and (C4) allows a "trade-off" between weaker moment bounds and "slower" mixing rates. Clearly, by the fact $\alpha(\tau) \leqslant 2$ and condition (C4), we have for all $n = 1, 2, 3, 4$,

$$B_n' \triangleq \int_0^{\infty} \tau^{n-1} [\alpha(\tau)]^{\delta/(\delta+2)}\, d\tau < \infty$$

and (2.5)

$$B_n \triangleq \int_0^{\infty} \tau^{n-1} [\alpha(\tau)]\, d\tau < \infty.$$

These constants will be used throughout later proofs.

(C5) For each $x \in \mathfrak{R}^d$, the limit

$$\bar{F}(x) \triangleq \lim_{T \to \infty} \frac{1}{T} \int_0^T EF(x, t)\, dt \tag{2.6}$$

exists. By (2.1) and (2.6), it follows that $x \to \bar{F}(x)$ has a global Lipschitz constant $N \triangleq d\bar{N}$. Let $x^0(\cdot)$ be the unique solution on $0 \leqslant \tau \leqslant 1$ of the equation

$$\dot{x}(\tau) = \bar{F}(x(\tau)) \qquad \text{subject to} \quad x(0) = x_0. \tag{2.7}$$

For convenience in later proofs, we define

$$D \triangleq \sup_{0 \leqslant \tau \leqslant 1} |x^0(\tau)|, \tag{2.8}$$

$$\tilde{F}(x, t) \triangleq F(x, t) - EF(x, t). \tag{2.9}$$

(C6) There exist constants $0<\chi\leqslant 1$ and $\gamma>0$, and some function $A(\cdot)$ such that

$$\sup_{\substack{t_0\geqslant 0\\ |x|\leqslant D}}\left|\frac{1}{T}\int_{t_0}^{t_0+T}\int_{t_0}^{t_0+T} E\{\tilde{F}_i(x,t)\tilde{F}_j(x,s)\}\,ds\,dt - A_{i,j}(x)\right| \leqslant \gamma T^{-\chi}$$
$$\text{for all}\quad T>0 \tag{2.10}$$

for all $1\leqslant i, j\leqslant d$, D being defined in (2.8).

(C7) There is some constant $c>0$ such that, for all $\varepsilon>0$ and $1\leqslant i, j\leqslant d$,

$$\sup_{0\leqslant\tau\leqslant 1}\left|\int_0^\tau EF_i(x^0(s), s/\varepsilon)-\bar{F}_i(x^0(s))\,ds\right|\leqslant c\varepsilon \tag{2.11}$$

and

$$\sup_{0\leqslant\tau\leqslant 1}\left|\int_0^\tau E\frac{\partial F_i}{\partial x_j}(x^0(s), s/\varepsilon)-\frac{\partial \bar{F}_i}{\partial x_j}(x^0(s))\,ds\right|\leqslant c\varepsilon. \tag{2.12}$$

Remark 2.1. Comparing our basic conditions (C0) to (C7) with the conditions in [17] which pertain to Khas'minskii's functional central limit theorem we see the following:

1. (C0) and (C3) are essentially (1.2) and the first part of (3.1), respectively, in [17], while (C1) is the second part of condition (3.1) in [17]. (C2) and (C4) are similar to condition (3.3) of [17], the difference being that our mixing rate is somewhat faster. Finally, (C5) is somewhat weaker than the first part of condition (3.2) in [17].

2. (C6) is stronger than the second part of condition (3.2) of [17], in that in [17] it is required only that the left-hand side of our (2.10) converge to 0 as $T\to\infty$, whereas we postulate a polynomial rate for this convergence. Actually, the stronger condition (C6) is satisfied by all examples considered in Khas'minskii [17]. This follows from Appendix 1 where it is shown that (C6) holds when $\{F(x,t),\ t\geqslant 0\}$ is weakly stationary (i.e., at each x, $EF(x,t)=EF(x,0)$ and $E\{F(x,t)F(x,s)\}=E\{F(x,0)\,F(x,t-s)\}$ for all $0\leqslant s\leqslant t$) and, more generally, when the functions $t\to EF(x,t)$ and $(t,s)\to E\{F(x,t)F(x,s)\}$ are periodic with period independent of x. As noted in [17] there are several interesting applications where these conditions on $F(x,t)$ are valid.

3. Condition (C7) is the same as condition (3.4) in Khas'minskii [17]. Again, it is shown in Appendix 1 that (C7) holds when, for example, $\{F(x,t),\ t\geqslant 0\}$ is weakly stationary or weakly periodic in the sense of (2) above.

In preparation for the statement and development of the central limit theorem with rate of convergence in Section 3, we introduce the following definitions. First, we define the process $\{Y^\varepsilon(\tau, \omega), 0 \leqslant \tau \leqslant 1\}$ on the original probability space $(\Omega, \mathscr{F}, P)$ by

$$Y^\varepsilon(\tau) \triangleq \varepsilon^{-1/2}(X^\varepsilon(\tau) - x^0(\tau)) \qquad \text{for all} \quad 0 \leqslant \tau \leqslant 1, \quad 0 < \varepsilon \leqslant 1. \tag{2.13}$$

Now suppose $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$ is a second probability space carrying some standard $\mathfrak{R}^d$-valued Brownian motion $\{\hat{B}^0(\tau, \hat{\omega}), 0 \leqslant \tau \leqslant 1\}$, and define the Gauss–Markov processes $\{\hat{W}^0(\tau, \hat{\omega}), 0 \leqslant \tau \leqslant 1\}$ and $\{\hat{Y}^0(\tau, \hat{\omega}), 0 \leqslant \tau \leqslant 1\}$ on $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$ by

$$d\hat{W}^0(\tau) \triangleq A^{1/2}(x^0(\tau))\, d\hat{B}^0(\tau) \qquad \text{subject to} \quad \hat{W}^0(0) = 0, \tag{2.14}$$

$$d\hat{Y}^0(\tau) \triangleq \frac{\partial \bar{F}}{\partial x}(x^0(\tau))\, \hat{Y}^0(\tau)\, d\tau + d\hat{W}^0(\tau) \qquad \text{subject to} \quad \hat{Y}^0(0) = 0, \tag{2.15}$$

where $A(\cdot) = (A^{1/2}(x))(A^{1/2}(x))^T$ is non-negative definite by (2.10) and $\bar{F}(\cdot)$ is given in (2.6).

3. Functional CLT with Error

We let $\Pi(Q_1, Q_2)$ be the Prohorov distance between two probability measures Q_1 and Q_2 defined on the Borel σ-algebra in $C[0, 1]$, the Banach space of $\mathfrak{R}^d$-valued continuous functions defined over $0 \leqslant \tau \leqslant 1$ with the norm

$$\|\psi\|_C \triangleq \max_{\substack{i = 1, \ldots, d \\ 0 \leqslant \tau \leqslant 1}} |\psi_i(\tau)|.$$

An assortment of useful facts pertaining to the Prohorov metric is given in Appendix 2. If $\{Q_\tau, 0 \leqslant \tau \leqslant 1\}$ is a process defined on some probability space whose sample paths are in $C[0, 1]$ then $\mathscr{L}(Q)$ will always denote the distribution probability measure in the Borel σ-algebra of $C[0, 1]$ generated by $\{Q_\tau\}$.

The main result of this note is the following:

Proposition 1. *Under the conditions of Section* 2, *there exist constants* $1 \geqslant \varepsilon_0 > 0$, $C > 0$, *and* $\lambda > 0$ *such that*

$$\Pi(\mathscr{L}(Y^\varepsilon), \mathscr{L}(\hat{Y}^0)) \leqslant C\varepsilon^\lambda \qquad \textit{for all} \quad 0 < \varepsilon \leqslant \varepsilon_0,$$

where for each $\varepsilon > 0$, *random processes* $Y^\varepsilon(\cdot)$ *and* $\hat{Y}^0(\cdot)$ *are defined in* (2.13) *and* (2.15).

Remark 3.1. In the above statement λ can be taken to be $3\mu/152$, where μ is calculated in Lemma A6.2 to be $\mu=\min\{1/33, \chi/16\}$ (χ being the constant of condition (C6) Section 2).

Proof of Proposition 1. Without loss of generality, we will assume that the P-null sets Λ_1 and Λ_2 in (C0) and (C1) are empty. Define the following processes on $(\Omega, \mathscr{F}, P)$ for each $\varepsilon>0$:

$$W_1^\varepsilon(\tau) \triangleq \varepsilon^{-1/2}\int_0^\tau \tilde{F}(x^0(s), s/\varepsilon)\, ds \qquad \text{for all} \quad 0\leqslant\tau\leqslant 1 \tag{3.1}$$

(where $\tilde{F}(\cdot,\cdot)$ is defined in Eq. (2.9)) and

$$Z_1^\varepsilon(\tau) \triangleq W_1^\varepsilon(\tau)+\int_0^\tau E\left[\frac{\partial F}{\partial x}(x^0(s), s/\varepsilon)\right] Z_1^\varepsilon(s)\, ds \qquad \text{for all } 0\leqslant\tau\leqslant 1. \tag{3.2}$$

Furthermore, we define a system of Gauss–Markov processes $\hat{Z}_2^\varepsilon(\cdot)$ on $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$ for each $\varepsilon>0$ by

$$\hat{Z}_2^\varepsilon(\tau) \triangleq \hat{W}^0(\tau)+\int_0^\tau E\left[\frac{\partial F}{\partial x}(x^0(s), s/\varepsilon)\right] \hat{Z}_2^\varepsilon(s)\, ds \qquad \text{for all} \quad 0\leqslant\tau\leqslant 1, \tag{3.3}$$

where $\hat{W}^0(\cdot)$ is defined in Eq. (2.14). Now, by the triangle inequality,

$$\Pi(\mathscr{L}(Y^\varepsilon), \mathscr{L}(\hat{Y}^0))\leqslant \Pi(\mathscr{L}(Y^\varepsilon), \mathscr{L}(Z_1^\varepsilon))+\Pi(\mathscr{L}(Z_1^\varepsilon), \mathscr{L}(\hat{Z}_2^\varepsilon)) + \Pi(\mathscr{L}(\hat{Z}_2^\varepsilon), \mathscr{L}(\hat{Y}^0)) \quad \text{for all} \quad \varepsilon>0. \tag{3.4}$$

In the remainder of the proof we bound each of the terms on the right of (3.4):

(a) Bound on $\Pi(\mathscr{L}(Y^\varepsilon), \mathscr{L}(Z_1^\varepsilon))$. Fix $0<\varepsilon\leqslant 1$ (to remain fixed throughout the proof of this bound), $\omega\in\Omega$, and $\tau\in[0, 1]$. Define

$$U_1^\varepsilon(\tau) \triangleq Y^\varepsilon(\tau)-Z_1^\varepsilon(\tau). \tag{3.5}$$

By (3.5), (2.13), (3.2), (3.1), and (2.9) (as well as (2.3) and (2.7)),

$$U_1^\varepsilon(\tau)=\frac{X^\varepsilon(\tau)-x^0(\tau)}{\sqrt{\varepsilon}}-\int_0^\tau \frac{\tilde{F}(x^0(s), s/\varepsilon)}{\varepsilon^{1/2}}\, ds - \int_0^\tau E\left[\frac{\partial F}{\partial x}(x^0(s), s/\varepsilon)\right] Z_1^\varepsilon(s)\, ds$$

$$= \frac{1}{\sqrt{\varepsilon}} \int_0^\tau (F(X^\varepsilon(s), s/\varepsilon) - F(x^0(s), s/\varepsilon))\, ds$$

$$- \int_0^\tau E\left[\frac{\partial F}{\partial x}(x^0(s), s/\varepsilon)\right] Z_1^\varepsilon(s)\, ds$$

$$+ \frac{1}{\sqrt{\varepsilon}} \left\{\int_0^\tau (EF(x^0(s), s/\varepsilon) - \bar{F}(x^0(s)))\, ds\right\}. \tag{3.6}$$

Now from (2.13), $X^\varepsilon(\tau) = x^0(\tau) + \sqrt{\varepsilon}\, Y^\varepsilon(\tau)$ so substituting this into (3.6),

$$U_1^\varepsilon(\tau) = \int_0^\tau \frac{\partial F}{\partial x}(x^0(s), s/\varepsilon)\, U_1^\varepsilon(s)\, ds + \int_0^\tau \Psi(s, \varepsilon)\, ds$$

$$+ I_1^\varepsilon(\tau) + \frac{1}{\sqrt{\varepsilon}} \int_0^\tau \{EF(x^0(s), s/\varepsilon) - \bar{F}(x^0(s))\}\, ds,$$

where

$$\Psi(s, \varepsilon) \triangleq \frac{1}{\sqrt{\varepsilon}} \left[F\left(x^0(s) + \sqrt{\varepsilon}\; Y^\varepsilon(s), \frac{s}{\varepsilon}\right) - F\left(x^0(s), \frac{s}{\varepsilon}\right)\right.$$

$$\left. - \frac{\partial F}{\partial x}\left(x^0(s), \frac{s}{\varepsilon}\right) \sqrt{\varepsilon}\; Y^\varepsilon(s)\right] \tag{3.7}$$

and

$$I_1^\varepsilon(\tau) \triangleq \int_0^\tau \left[\frac{\partial F}{\partial x}(x^0(s), s/\varepsilon) - E\frac{\partial F}{\partial x}(x^0(s), s/\varepsilon)\right] Z_1^\varepsilon(s)\, ds. \tag{3.8}$$

Therefore by (3.7), (2.1), and (2.11) (recall $N \triangleq d\bar{N}$),

$$|U_1^\varepsilon(\tau)| \leqslant N \int_0^\tau |U_1^\varepsilon(s)|\, ds + \int_0^1 |\Psi(s, \varepsilon)|\, ds + \|I_1^\varepsilon\|_C + c\varepsilon^{1/2};$$

hence, by the Gronwall inequality (see, e.g., Lemma 1.1, Chap. 2 of Freidlin and Wentzell [11]),

$$|U_1^\varepsilon(\tau)| \leqslant e^{N\tau} \left\{\int_0^1 |\Psi(s, \varepsilon)|\, ds + \|I_1^\varepsilon\|_C + c\varepsilon^{1/2}\right\} \tag{3.9}$$

for all $0 \leqslant \tau \leqslant 1$ and $\omega \in \Omega$, so by (3.9) and Hölder's inequality,

$$E\,\|U_1^\varepsilon\|_C \leqslant e^N \left\{E\int_0^1 |\Psi(s, \varepsilon)|\, ds + (E\,\|I_1^\varepsilon\|_C^4)^{1/4} + c\varepsilon^{1/2}\right\}. \tag{3.10}$$

Now by condition (C1) and Taylor's formula for each $s \in [0, 1]$ and $\omega \in \Omega$,

$$\begin{aligned} F(x^0(s) + \sqrt{\varepsilon}\, Y^\varepsilon(s), s/\varepsilon) &\\ = F(x^0(s), s/\varepsilon) + \frac{\partial F}{\partial x}(x^0(s), s/\varepsilon)(\sqrt{\varepsilon}\, Y^\varepsilon(s)) & \qquad (3.11) \\ + \frac{1}{2}\varepsilon \sum_{j,k=1}^{d} \int_0^1 (1-\zeta) \frac{\partial^2 F}{\partial x_j\, \partial x_k}(x^0(s) + \zeta \varepsilon^{1/2} Y^\varepsilon(s), s/\varepsilon)\, d\zeta\, Y_j^\varepsilon(s) Y_k^\varepsilon(s). & \end{aligned}$$

By (3.7), (3.11), and (2.2),

$$\begin{aligned} |\Psi(s, \varepsilon)| &= \frac{1}{2\varepsilon^{1/2}} \left| \varepsilon \sum_{j,k=1}^{d} \int_0^1 (1-\zeta) \frac{\partial^2 F}{\partial x_j\, \partial x_k}(x^0(s) \right. \\ &\qquad \left. + \zeta \varepsilon^{1/2} Y^\varepsilon(s), s/\varepsilon)\, d\zeta\, Y_j^\varepsilon(s) Y_k^\varepsilon(s) \right| \\ &\leqslant \frac{1}{2}\varepsilon^{1/2} dN \|Y^\varepsilon\|_C^2 \qquad \text{for all} \quad 0 \leqslant s \leqslant 1 \quad \text{and} \quad \omega \in \Omega. \qquad (3.12) \end{aligned}$$

Now by Lemma A5.1 of Appendix 5 there is some constant $c_1 > 0$ (depending only on constants M, N, D, d, and B_1', B_2' of Section 2) such that

$$E[\max_{0 \leqslant s \leqslant 1} |Y^\varepsilon(s)|^2] \leqslant c_1 \qquad \text{for all} \quad \varepsilon > 0. \qquad (3.13)$$

In view of (3.12) and (3.13),

$$E \int_0^1 |\Psi(s, \varepsilon)|\, ds \leqslant \varepsilon^{1/2} dN c_1. \qquad (3.14)$$

To bound $E\,\|I_1^\varepsilon\|_C^4$ in Eq. (3.10) define the d by d matrix-valued function $H^\varepsilon(\tau)$, $0 \leqslant \tau \leqslant 1$, to be the solution of the matrix differential equation

$$\dot{H}^\varepsilon(\tau) = E\left[\frac{\partial F}{\partial x}(x^0(\tau), \tau/\varepsilon)\right] H^\varepsilon(\tau) \qquad \text{subject to} \quad H^\varepsilon(0) = I. \quad (3.15)$$

By the theory of linear ordinary matrix differential equations (see, e.g., the background theory on pages 253–255 in Kallianpur [15]), $H^\varepsilon(\cdot)$ is unique, $H^\varepsilon(\tau)$ is nonsingular for all $0 \leqslant \tau \leqslant 1$, and

$$\frac{d[H^\varepsilon(\tau)]^{-1}}{d\tau} = -[H^\varepsilon(\tau)]^{-1} E\left[\frac{\partial F(x^0(\tau), \tau/\varepsilon)}{\partial x}\right]. \qquad (3.16)$$

Thus from (3.2), Eqs. (10.2.2b) and (10.2.5) in Kallianpur [15, p. 255], and integration by parts, we obtain

$$Z_1^\varepsilon(\tau) = W_1^\varepsilon(\tau) + \int_0^\tau K^\varepsilon(\tau, u)\, W_1^\varepsilon(u)\, du \quad \text{for all} \quad 0 \leqslant \tau \leqslant 1, \quad \omega \in \Omega, \tag{3.17}$$

where

$$K^\varepsilon(\tau, u) \triangleq H^\varepsilon(\tau)[H^\varepsilon(u)]^{-1} E\left[\frac{\partial F}{\partial x}(x^0(u), u/\varepsilon)\right] \quad \text{for} \quad 0 \leqslant \tau, u \leqslant 1. \tag{3.18}$$

(For later use in Appendix 5 we note that with modest work one can establish from (3.15), (2.1), and two applications of the Gronwall inequality that

$$\max_{1 \leqslant i, j \leqslant d} |K_{ij}^\varepsilon(\tau, u)| \leqslant N e^{2N} \tag{3.19}$$

for all $\varepsilon > 0$, $0 \leqslant \tau$, $u \leqslant 1$.)

Now fix $\tau \in [0, 1]$ and $\omega \in \Omega$. By (3.8) and (3.17),

$$I_1^\varepsilon(\tau) = A^\varepsilon(\tau) + B^\varepsilon(\tau), \tag{3.20}$$

where for all $v \in [0, 1]$,

$$A^\varepsilon(v) \triangleq \int_0^v \left\{\frac{\partial F}{\partial x}(x^0(s), s/\varepsilon) - E\left[\frac{\partial F}{\partial x}(x^0(s), s/\varepsilon)\right]\right\} W_1^\varepsilon(s)\, ds \tag{3.21}$$

and

$$B^\varepsilon(v) \triangleq \int_0^v \left\{\frac{\partial F}{\partial x}(x^0(s), s/\varepsilon) - E\left[\frac{\partial F}{\partial x}(x^0(s), s/\varepsilon)\right]\right\} \times \int_0^s K^\varepsilon(s, u)\, W_1^\varepsilon(u)\, du\, ds. \tag{3.22}$$

Therefore,

$$|I_1^\varepsilon(\tau)|^4 \leqslant 8\, |A^\varepsilon(\tau)|^4 + 8\, |B^\varepsilon(\tau)|^4 \quad \text{for all} \quad 0 \leqslant \tau \leqslant 1 \quad \text{and} \quad \omega \in \Omega, \tag{3.23}$$

so

$$E\, \|I_1^\varepsilon\|_C^4 \leqslant 8E\{\|A^\varepsilon\|_C^4\} + 8E\{\|B^\varepsilon\|_C^4\}. \tag{3.24}$$

By (3.24), Lemma A5.2, and Lemma A5.3, there exists a constant $c_2 > 0$, depending only on constants M, $N \triangleq d\bar{N}$, D, d, and $B'_1, ..., B'_4$ of Section 2, such that

$$E \|I_1^\varepsilon\|_C^4 \leqslant c_2 \varepsilon^2. \tag{3.25}$$

Thus by (3.5), (3.10), (3.14), and (3.25) there exists a constant $c_3 > 0$, depending only on M, N, D, d, and $B'_1, ..., B'_4$, such that

$$E\{\|Y^\varepsilon - Z_1^\varepsilon\|_C\} = E\|U_1^\varepsilon\|_C \leqslant c_3 \varepsilon^{1/2}. \tag{3.26}$$

Thus releasing ε, we have by Lemma A2.2(b) of Appendix 2

$$\Pi(\mathscr{L}(Y^\varepsilon), \mathscr{L}(Z_1^\varepsilon)) \leqslant \sqrt{c_3}\, \varepsilon^{1/4} \qquad \text{for} \quad 0 < \varepsilon \leqslant 1. \tag{3.27}$$

This establishes a bound on the first term on the right-hand side of (3.4).

(b) Bound on $\Pi(\mathscr{L}(Z_1^\varepsilon), \mathscr{L}(\hat{Z}_2^\varepsilon))$. In order to bound the second term on the right-hand side of (3.4), we first consider $\Pi(\mathscr{L}(W_1^\varepsilon), \mathscr{L}(\hat{W}^0))$, where $W_1^\varepsilon(\cdot)$ and $\hat{W}^0(\cdot)$ are defined by (3.1) and (2.14), respectively. By Lemma A6.1 of Appendix 6 there exist constants $c_4 > 0$, $\rho > 0$, and $0 < \varepsilon_0 \leqslant 1$, such that

$$\Pi(\mathscr{L}(W_1^\varepsilon), \mathscr{L}(\hat{W}^0)) < c_4 \varepsilon^\rho \qquad \text{for all} \quad 0 < \varepsilon \leqslant \varepsilon_0. \tag{3.28}$$

In view of (3.28) and the Strassen–Dudley theorem (see Lemma A2.1 of Appendix 2), for each $\varepsilon \in (0, \varepsilon_0]$ there is some probability measure $\tilde{P}^\varepsilon$ on $(\tilde{\Omega}, \tilde{\mathscr{F}}) \equiv (C[0,1] \times C[0,1], \mathscr{B}(C[0,1] \times C[0,1]))$, where

$$\tilde{P}^\varepsilon\{(x, y) : \|x - y\|_C > c_4 \varepsilon^\rho\} \leqslant c_4 \varepsilon^\rho \tag{3.29}$$

and $\mathscr{B}(C[0,1] \times C[0,1]))$ denotes the Borel sets in the product topology of $C[0,1] \times C[0,1]$. Moreover, the marginals of $\tilde{P}^\varepsilon$ on $(C[0,1], \mathscr{B}(C[0,1]))$ are $\mathscr{L}(W_1^\varepsilon)$ and $\mathscr{L}(\hat{W}^0)$.

Now fix an $\varepsilon \in (0, \varepsilon_0]$ and for each $\tilde{\omega} \triangleq (\tilde{\omega}_1, \tilde{\omega}_2) \in \tilde{\Omega}$ put

$$\tilde{W}_1^\varepsilon(\tau, \tilde{\omega}) \triangleq \tilde{\omega}_1(\tau) \quad \text{and} \quad \tilde{W}^0(\tau, \tilde{\omega}) \triangleq \tilde{\omega}_2(\tau) \quad \text{for all} \quad 0 \leqslant \tau \leqslant 1. \tag{3.30}$$

Then $\{\tilde{W}_1^\varepsilon(\tau), 0 \leqslant \tau \leqslant 1\}$ and $\{\tilde{W}^0(\tau), 0 \leqslant \tau \leqslant 1\}$ are $\mathfrak{R}^d$-value stochastic processes on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}^\varepsilon)$, $\mathscr{L}(\tilde{W}_1^\varepsilon) = \mathscr{L}(W_1^\varepsilon)$, $\mathscr{L}(\tilde{W}^0) = \mathscr{L}(\hat{W}^0)$, and by (3.29),

$$\tilde{P}^\varepsilon(\|\tilde{W}_1^\varepsilon - \tilde{W}^0\|_C > c_4 \varepsilon^\rho) \leqslant c_4 \varepsilon^\rho. \tag{3.31}$$

Now fix $\tilde{\omega} \in \tilde{\Omega}$ and $\tau \in [0, 1]$ and define (comparing with (3.2) and (3.3))

$$\tilde{Z}_1^\varepsilon(\tau) \triangleq \tilde{W}_1^\varepsilon(\tau) + \int_0^\tau E\left[\frac{\partial F}{\partial x}(x^0(s), s/\varepsilon)\right] \tilde{Z}_1^\varepsilon(s)\, ds, \tag{3.32}$$

$$\tilde{Z}_2^\varepsilon(\tau) \triangleq \tilde{W}^0(\tau) + \int_0^\tau E\left[\frac{\partial F}{\partial x}(x^0(s), s/\varepsilon)\right] \tilde{Z}_2^\varepsilon(s)\, ds. \tag{3.33}$$

Thus by (2.1), (3.32), (3.33), and Gronwall, for each $\tilde{\omega} \in \tilde{\Omega}$,

$$\|\tilde{Z}_1^\varepsilon - \tilde{Z}_2^\varepsilon\|_C \leqslant e^N \|\tilde{W}_1^\varepsilon - \tilde{W}^0\|_C. \tag{3.34}$$

By (3.34) and (3.31),

$$\begin{aligned} \tilde{P}^\varepsilon\{\|\tilde{Z}_1^\varepsilon - \tilde{Z}_2^\varepsilon\|_C \geqslant c_4 e^N \varepsilon^\rho + c_4 \varepsilon^\rho\} &\leqslant \tilde{P}^\varepsilon\{\|\tilde{Z}_1^\varepsilon - \tilde{Z}_2^\varepsilon\|_C > c_4 e^N \varepsilon^\rho\} \\ &\leqslant \tilde{P}^\varepsilon\{\|\tilde{W}_1^\varepsilon - \tilde{W}^0\|_C > c_4 \varepsilon^\rho\} \\ &\leqslant c_4 \varepsilon^\rho < c_4 e^N \varepsilon^\rho + c_4 \varepsilon^\rho. \end{aligned} \tag{3.35}$$

Thus releasing ε we obtain by Lemma A2.2,

$$\Pi(\mathscr{L}(\tilde{Z}_1^\varepsilon), \mathscr{L}(\tilde{Z}_2^\varepsilon)) \leqslant c_4(e^N + 1)\varepsilon^\rho \qquad \text{for all} \quad 0 < \varepsilon \leqslant \varepsilon_0. \tag{3.36}$$

But from (3.32), (3.2), and the fact that $\mathscr{L}(W_1^\varepsilon) = \mathscr{L}(\tilde{W}_1^\varepsilon)$, we see $\mathscr{L}(Z_1^\varepsilon) = \mathscr{L}(\tilde{Z}_1^\varepsilon)$ for all $0 < \varepsilon \leqslant 1$. Similarly from (3.33), (3.3), and the fact that $\mathscr{L}(\hat{W}^0) = \mathscr{L}(\tilde{W}^0)$, we obtain $\mathscr{L}(\hat{Z}_2^\varepsilon) = \mathscr{L}(\tilde{Z}_2^\varepsilon)$ for all $0 < \varepsilon \leqslant 1$. Thus by (3.36),

$$\Pi(\mathscr{L}(Z_1^\varepsilon), \mathscr{L}(\hat{Z}_2^\varepsilon)) \leqslant c_4(e^N + 1)\varepsilon^\rho \qquad \text{for all} \quad 0 < \varepsilon \leqslant \varepsilon_0. \tag{3.37}$$

This establishes a bound on the second term on the right-hand side of (3.4).

(c) Bound on $\Pi(\mathscr{L}(\hat{Z}_2^\varepsilon), \mathscr{L}(\hat{Y}0))$. Consider the third term on the right-hand side of (3.4). We fix $\varepsilon \in (0, 1]$, $\hat{\omega} \in \hat{\Omega}$, and $\tau \in [0, 1]$, and put

$$\hat{U}_2^\varepsilon(\tau) \triangleq \hat{Z}_2^\varepsilon(\tau) - \hat{Y}^0(\tau). \tag{3.38}$$

Then by (3.3), (2.15), and (2.1),

$$\begin{aligned} |\hat{U}_2^\varepsilon(\tau)| &= \left| \int_0^\tau \left\{ E\left[\frac{\partial F}{\partial x}(x^0(s), s/\varepsilon)\right] \hat{Z}_2^\varepsilon(s) - \frac{\partial \bar{F}}{\partial x}(x^0(s))\, \hat{Y}^0(s) \right\} ds \right| \\ &\leqslant N \int_0^\tau |\hat{U}_2^\varepsilon(s)|\, ds + \max_{0 \leqslant \tau \leqslant 1} \left| \int_0^\tau \left\{ E \frac{\partial F}{\partial x}\left(x^0(s), \frac{s}{\varepsilon}\right) \right.\right. \\ &\qquad \left.\left. - \frac{\partial \bar{F}}{\partial x}(x^0(s)) \right\} \hat{Y}^0(s)\, ds \right| \end{aligned} \tag{3.39}$$

for all $0 \leqslant \tau \leqslant 1$. Hence by Gronwall,

$$\|\hat{U}_2^\varepsilon\|_C \leqslant e^N \max_{0 \leqslant \tau \leqslant 1} \left| \int_0^\tau \left\{ E\left[\frac{\partial F}{\partial x}(x^0(s), s/\varepsilon)\right] - \frac{\partial \bar{F}}{\partial x}(x^0(s)) \right\} \hat{Y}^0(s)\, ds \right|. \tag{3.40}$$

We now adapt to a stochastic setting an argument for ordinary differential equations which is due to Gihman [12, pp. 216–217]. Fix a positive integer n and define the process $\{\hat{Y}_n^0(\tau),\ 0 \leqslant \tau < 1\}$ by

$$\hat{Y}_n^0(\tau) \triangleq \hat{Y}^0(i/n) \qquad \text{for all } \frac{i}{n} \leqslant \tau < \frac{i+1}{n} \quad \text{and} \quad i = 0, 1, 2, ..., (n-1). \tag{3.41}$$

By (3.40) and (2.1),

$$\begin{aligned} \|\hat{U}_2^\varepsilon\|_C \leqslant e^N \max_{0 \leqslant \tau \leqslant 1} \Bigg\{ & \int_0^\tau \max_{1 \leqslant i \leqslant d} \sum_{j=1}^d \left| E\left[\frac{\partial F_i}{\partial x_j}(x^0(s), s/\varepsilon)\right]\right| \\ & \times |\hat{Y}^0(s) - \hat{Y}_n^0(s)|\, ds \\ & + \left| \int_0^\tau \left\{ E\left[\frac{\partial F}{\partial x}(x^0(s), s/\varepsilon)\right] - \frac{\partial \bar{F}}{\partial x}(x^0(s)) \right\} \hat{Y}_n^0(s)\, ds \right| \\ & + \int_0^\tau \max_{1 \leqslant i \leqslant d} \sum_{j=1}^d \left| \frac{\partial \bar{F}_i}{\partial x_j}(x^0(s)) \right| \cdot |\hat{Y}_n^0(s) - \hat{Y}^0(s)|\, ds \Bigg\} \\ \leqslant 2Ne^N \|\hat{Y}^0 - \hat{Y}_n^0\|_C & + e^N \max_{0 \leqslant \tau \leqslant 1} \left| \int_0^\tau \Gamma^\varepsilon(s) \hat{Y}_n^0(s)\, ds \right|, \end{aligned} \tag{3.42}$$

where

$$\Gamma^\varepsilon(s) \triangleq E\left[\frac{\partial F}{\partial x}(x^0(s), s/\varepsilon)\right] - \frac{\partial \bar{F}}{\partial x}(x^0(s)) \qquad \text{for all} \quad 0 \leqslant s \leqslant 1. \tag{3.43}$$

Now by (3.43) and (2.12) for any $0 \leqslant \tau_1, \tau_2 \leqslant 1$ and integers $i, j \in \{1, ..., d\}$,

$$\left| \int_{\tau_1}^{\tau_2} \Gamma_{i,j}^\varepsilon(s)\, ds \right| \leqslant \left| \int_0^{\tau_2} \Gamma_{i,j}^\varepsilon(s)\, ds \right| + \left| \int_0^{\tau_1} \Gamma_{i,j}^\varepsilon(s)\, ds \right| \leqslant 2c\varepsilon. \tag{3.44}$$

Consider the second term on the far right of (3.42) and let $|T|_\infty$ denote $\max_{1 \leqslant i,j \leqslant d} |T_{ij}|$ when T is a d by d matrix. We have by (3.41) and (3.44) that for any $0 \leqslant \tau \leqslant 1$,

$$\left|\int_0^\tau \Gamma^\varepsilon(s)\hat{Y}_n^0(s)\,ds\right| \leqslant \left|\sum_{i=0}^{[\tau n]-1}\int_{i/n}^{(i+1)/n} \Gamma^\varepsilon(s)\hat{Y}_n^0(s)\,ds\right|$$
$$+\left|\int_{[\tau n]/n}^{\tau} \Gamma^\varepsilon(s)\hat{Y}_n^0(s)\,ds\right|$$
$$\leqslant d\sum_{i=0}^{[\tau n]-1}\left|\int_{i/n}^{(i+1)/n} \Gamma^\varepsilon(s)\,ds\right|_\infty \cdot \left|\hat{Y}^0\left(\frac{i}{n}\right)\right|$$
$$+d\left|\int_{[\tau n]/n}^{\tau} \Gamma^\varepsilon(s)\,ds\right|_\infty \cdot \left|\hat{Y}^0\left(\frac{[\tau n]}{n}\right)\right|$$
$$\leqslant d\,\|\hat{Y}^0\|_C\left\{[\tau n](2c\varepsilon)+\left|\int_{[\tau n]/n}^{\tau} \Gamma^\varepsilon(s)\,ds\right|_\infty\right\}$$
$$\leqslant \|\hat{Y}^0\|_C\, 2\, dnc\varepsilon, \tag{3.45}$$

where the last inequality follows from (3.44) and the fact that $[\tau n]\leqslant n-1$ for $0\leqslant\tau<1$. Thus by (3.42) and (3.45),

$$\|\hat{U}_2^\varepsilon\|_C\leqslant 2Ne^N\,\|\hat{Y}^0-\hat{Y}_n^0\|_C+e^N 2\,dcn\varepsilon\,\|\hat{Y}^0\|_C \qquad \text{for all} \quad \hat{\omega}\in\hat{\Omega} \tag{3.46}$$

and, so for any $a>0$,

$$\hat{P}(\|\hat{U}_2^\varepsilon\|_C\geqslant a)\leqslant \hat{P}\left\{\|\hat{Y}^0-\hat{Y}_n^0\|_C\geqslant\frac{a}{4Ne^N}\right\}+\hat{P}\left\{\|\hat{Y}^0\|_C\geqslant\frac{a}{4de^N cn\varepsilon}\right\}. \tag{3.47}$$

But, since

$$\left\{\hat{\omega}:\|\hat{Y}^0-\hat{Y}_n^0\|_C\geqslant\frac{a}{4Ne^N}\right\}$$
$$\subset\bigcup_{i=0}^{n-1}\left\{\hat{\omega}:\max_{i/n\leqslant s\leqslant(i+1)/n}\left|\hat{Y}^0(s)-\hat{Y}^0\left(\frac{i}{n}\right)\right|\geqslant\frac{a}{4Ne^N}\right\}, \tag{3.48}$$

we have by the Chebyshev inequality,

$$\hat{P}(\|\hat{U}_2^\varepsilon\|_C\geqslant a)$$
$$\leqslant \hat{P}\left\{\|\hat{Y}^0\|_C\geqslant\frac{a}{4de^N cn\varepsilon}\right\}$$
$$+\sum_{i=0}^{n-1}\hat{P}\left(\max_{i/n\leqslant s\leqslant(i+1)/n}|\hat{Y}^0(s)-\hat{Y}^0(i/n)|\geqslant\frac{a}{4Ne^N}\right)$$
$$\leqslant \hat{E}\big(\max_{0\leqslant\tau\leqslant 1}|\hat{Y}^0(\tau)|^2\big)\,a^{-2}(4de^N cn\varepsilon)^2$$
$$+\sum_{i=0}^{n-1}\hat{E}\big(\max_{i/n\leqslant s\leqslant(i+1)/n}|\hat{Y}^0(s)-\hat{Y}^0(i/n)|^4\big)a^{-4}(4Ne^N)^4. \tag{3.49}$$

Also, by (2.6) and (2.1),

$$\left|\frac{\partial \bar{F}_i}{\partial x_j}(x^0(\tau))\right| \leqslant \bar{N} \qquad \text{for all} \quad 0 \leqslant \tau \leqslant 1, \quad i, j \in \{1, ..., d\} \tag{3.50}$$

and, by Lemma A7.1 and the continuity of the function $\tau \to x^0(\tau)$, there exists a $M' > 0$ such that

$$|A_{i,j}^{1/2}(x^0(\tau))| \leqslant M' \qquad \text{for all} \quad 0 \leqslant \tau \leqslant 1, \quad i, j \in \{1, ..., d\}. \tag{3.51}$$

Hence by (2.14), (2.15), and (5.3.18) of [16, p. 306] there exists some $c_6 > 0$ such that

$$\hat{E}\,|\hat{Y}^0(\tau) - \hat{Y}^0(\tau')|^4 \leqslant c_6\,|\tau - \tau'|^2 = (h(\tau', \tau))^2 \quad \text{for all } 0 \leqslant \tau' \leqslant \tau \leqslant 1, \tag{3.52}$$

where

$$h(\tau', \tau) \triangleq c_6^{1/2}(\tau - \tau') \qquad \text{for all} \quad 0 \leqslant \tau' \leqslant \tau \leqslant 1. \tag{3.53}$$

Thus by Theorem A3.1 of Appendix 3 (with $\nu \triangleq 4$, $\gamma \triangleq 2$, $Q_t \triangleq \hat{Y}^0(t)$) there exists $c_7 > 0$ such that

$$\hat{E}(\max_{i/n \leqslant s \leqslant (i+1)/n} |\hat{Y}^0(s) - \hat{Y}^0(i/n)|^4) \leqslant c_7 n^{-2} \quad \text{for all} \quad i = 0, 1, ..., (n-1). \tag{3.54}$$

Moreover, by (3.50), (3.51), and (5.3.17) of [16, p. 306] there exists constant $c_8 > 0$ such that

$$\hat{E}(\max_{0 \leqslant \tau \leqslant 1} |\hat{Y}^0(\tau)|^2) \leqslant c_8. \tag{3.55}$$

Thus by (3.49), (3.54), and (3.55) there is some $c_9 > 0$ such that

$$\hat{P}(\|\hat{U}_2^\varepsilon\|_C \geqslant a) \leqslant c_9\{a^{-2}n^2\varepsilon^2 + a^{-4}n^{-1}\} \tag{3.56}$$

for any $a > 0$ and positive integer n. Now by (3.56) there exists some $c_{10} > 0$ such that if we define $a(\varepsilon) \triangleq c_{10}\varepsilon^{1/8}$, $n(\varepsilon) \triangleq [\varepsilon^{-2/3}]$ then

$$\hat{P}(\|\hat{U}_2^\varepsilon\|_C \geqslant c_{10}\varepsilon^{1/8}) \leqslant c_9\left\{\frac{1}{c_{10}^2}\varepsilon^{5/12} + \frac{2}{c_{10}^4}\varepsilon^{1/6}\right\} < c_{10}\varepsilon^{1/8}. \tag{3.57}$$

Now using Lemma A2.2 and releasing ε,

$$\Pi(\mathscr{L}(\hat{Z}_2^\varepsilon), \mathscr{L}(\hat{Y}^0)) \leqslant c_{10}\varepsilon^{1/8} \qquad \text{for all} \quad 0 < \varepsilon \leqslant 1. \tag{3.58}$$

Proposition 1 follows from (3.4), (3.27), (3.37), and (3.58). ∎

APPENDIX 1: Applicability of Conditions (C5), (C6), (C7)

In this appendix we establish a lemma which illustrates the applicability of conditions (C5), (C6) and (C7) in Section 2. A particular consequence of this lemma is that these conditions are satisfied by all the examples considered on pages 222 to 227 of Khas'minskii [17]. We shall call the system of processes $\{F(x, t, \omega),\ t \geqslant 0\}$ *wide-sense periodic with periodicity* $\theta > 0$ *if*: (a) $EF(x, t) = EF(x, t + \theta)$ and (b) $E\{\tilde{F}(x, t)(\tilde{F}(x, s))^T\} = E\{\tilde{F}(x, t+\theta)(\tilde{F}(x, s+\theta))^T\}$ for all $x \in \Re^d$, $s, t \geqslant 0$. Without loss of generality we shall assume that the function $\Gamma(x, t, s)$, defined by $\Gamma(x, t, s) \triangleq E\{\tilde{F}(x, t)(\tilde{F}(x, s))^T\}$ for $s, t \geqslant 0$, has been uniquely extended over all $-\infty < s,\ t < \infty$ by the periodicity relation $\Gamma(x, t+\theta,\ s+\theta) = \Gamma(x, t, s)$ for all x, s, t.

Lemma A1.1. *Suppose that* $\{F(x, t, \omega),\ t \geqslant 0\}$ *satisfies conditions* (C0) *to* (C4) *in Section* 2 *and is also wide-sense periodic with periodicity* θ. *Then* (C5), (C6), *and* (C7) *of Section* 2 *hold with* $\bar{F}(x)$ *in* (2.6) *and* $A(x)$ *in* (2.10) *given by*

$$\bar{F}(x) \triangleq \frac{1}{\theta} \int_0^\theta EF(x, t)\, dt, \tag{A1.1}$$

$$A(x) \triangleq \frac{1}{\theta} \int_0^\theta \int_{-\infty}^{\infty} \Gamma(x, t, s)\, ds\, dt, \tag{A1.2}$$

the integral on the right of (A1.2) *is well defined, and* χ *in* (2.10) *is given by* $\chi \triangleq 1$.

Remark A1.1. This lemma was motivated by Eqs. (3.23) and (3.24) on page 222 of Khas'minskii [17]. In contrast to the treatment in [17], we do not require Hölder continuity of the function $t \to EF(x^0(t), t)$ ($x^0(\cdot)$ given by (2.7)) or make use of Fourier analysis.

Remark A1.2. In the special case where $\{F(x, t, \omega),\ t \geqslant 0\}$ is *wide-sense stationary* for each x (i.e., $EF(x, s) = EF(x, 0)$ and $E\{F(x, s)(F(x, t))^T\} = E\{F(x, 0)(F(x, t-s))^T\}$ for all $x \in \Re^d$, $0 \leqslant s \leqslant t$) and satisfies conditions (C0) to (C4) of Section 2, it follows from Lemma A1.1 that (C5), (C6), (C7) hold with $\bar{F}(x) \triangleq EF(x, 0)$, $A(x) \triangleq \int_0^\infty E\{\tilde{F}(x, 0)(\tilde{F}(x, t))^T + \tilde{F}(x, t)(\tilde{F}(x, 0))^T\}\, dt$, $\chi \triangleq 1$.

Proof of Lemma A1.1. (i) Fix some $x \in \Re^d$. By the θ-periodicity of $t \to EF(x, t)$, and (A1.1):

$$\left|\frac{1}{T}\int_0^T EF(x,t)\,dt - \bar{F}(x)\right| \leqslant \left|\left(\frac{[T/\theta]}{T} - \frac{1}{\theta}\right)\int_0^\theta EF(x,t)\,dt\right| + \left|\frac{1}{T}\int_{[T/\theta]\theta}^T EF(x,t)\,dt\right|. \tag{A1.3}$$

(C5) follows upon taking $T \to \infty$ in (A1.3) and noting that $\sup_{t \geqslant 0} |EF(x, t)| < \infty$ (see Lemma A7.2(i)).

(ii) We prove only (2.12) of (C7), since the proof of (2.11) is similar but easier. From (A1.1) we obtain $\int_0^\theta (EF(x, s) - \bar{F}(x))\,ds = 0$ for all x and, hence, by (2.1) and the dominated convergence theorem,

$$\int_0^\theta \left(E\frac{\partial F}{\partial x}(x,s) - \frac{\partial \bar{F}}{\partial x}(x)\right) ds = 0. \tag{A1.4}$$

Moreover, it follows from θ-periodicity of $t \to EF(x, t)$ and the dominated convergence theorem that $t \to E((\partial F/\partial x)(x, t)) - (\partial \bar{F}/\partial x)(x)$ is θ-periodic. Defining $G(x) \triangleq \bar{F}(x)$ and $Z(x, t) \triangleq E((\partial F_i/\partial x_j)(x, t)) - (\partial \bar{F}_i/\partial x_j)(x)$, we see that (2.12) is an immediate consequence of the following special case of a lemma due to Besjes [3, Lemma 1, p. 362]:

LEMMA. *Suppose that $\eta(\cdot)$ is some solution, defined everywhere over the closed unit interval $0 \leqslant \tau \leqslant 1$, of the differential equation in $\mathfrak{R}^d$,*

$$\dot{\eta}(\tau) = G(\eta(\tau)) \qquad \text{subject to} \quad \eta(0) = \eta_0,$$

where $G(\cdot)$ is continuous. If $Z(x, t)$ is a real-valued function defined for all $t \geqslant 0$, $x \in \mathfrak{R}^d$, such that

(a) *for some constant Λ, one has $|Z(x, t) - Z(x', t)| \leqslant \Lambda\,|x - x'|$ for all x, $x' \in \mathfrak{R}^d$ and $t \geqslant 0$;*

(b) *for each x, $t \to Z(x, t)$ is measurable and θ-periodic; and*

(c) *$\int_0^\theta Z(x, t)\,dt = 0$ for each x;*

then there is an absolute constant c such that $\sup_{0 \leqslant \tau \leqslant 1} |\int_0^\tau Z(\eta(s), s/\varepsilon)\,ds| \leqslant c\varepsilon$ for all $\varepsilon > 0$.

(iii) It remains to prove (C6) with $A(x)$ given by (A1.2). From (C2), Lemma A3.2, and Lemma A7.2(ii),

$$|\Gamma_{i,j}(x, t, s)| \leqslant 40(M + N\,|x|)^2\,\{\alpha(|t - s|)\}^{\delta/(2+\delta)} \tag{A1.5}$$

for all $x \in \Re^d$, $1 \leqslant i, j \leqslant d$, and $s, t \geqslant 0$. Thus, for all $t_1 \geqslant 0$, $T \geqslant 0$,

$$\begin{aligned}\int_{t_1}^{t_1+T} \int_{-\infty}^{\infty} |\Gamma_{i,j}(x, t, s)| \, ds \, dt &\leqslant 40(M + N|x|)^2 \int_{t_1}^{t_1+T} \int_{-\infty}^{\infty} \{\alpha(|t-s|)\}^{\delta/(\delta+2)} \, ds \, dt \\ &= 80(M+N|x|)^2 \, T \int_0^{\infty} \{\alpha(v)\}^{\delta/(\delta+2)} \, dv \\ &\leqslant 80(M+N|x|)^2 \, TB_1'. \end{aligned} \tag{A1.6}$$

Clearly $A(x)$ is well defined and from the θ-periodicity of $(s, t) \to \Gamma(x, t, s)$ it follows that

$$A(x) = \theta^{-1} \int_{t_1}^{t_1+\theta} \int_{-\infty}^{\infty} \Gamma(x, t, s) \, ds \, dt \qquad \text{for all} \quad t_1 \geqslant 0. \tag{A1.7}$$

Thus, for all x, $t_0 \geqslant 0$, $T > 0$,

$$\begin{aligned}\frac{1}{T} \int_{t_0}^{t_0+T} \int_{-\infty}^{\infty} \Gamma(x, t, s) \, ds \, dt \\ = \frac{\theta}{T} \left[\frac{T}{\theta} \right] A(x) + \frac{1}{T} \int_{t_0+\theta[T/\theta]}^{t_0+T} \int_{-\infty}^{\infty} \Gamma(x, t, s) \, ds \, dt. \end{aligned} \tag{A1.8}$$

Now from (A1.5),

$$\begin{aligned}\int_{t_0}^{\infty} \int_{-\infty}^{t_0} |\Gamma_{i,j}(x, t, s)| \, ds \, dt &\leqslant 40(M+N|x|)^2 \int_0^{\infty} v \{\alpha(v)\}^{\delta/(\delta+2)} \, dv \\ &\leqslant 40(M+N|x|)^2 \, B_2' \end{aligned} \tag{A1.9}$$

and, similarly,

$$\int_{-\infty}^{t_0+T} \int_{t_0+T}^{\infty} |\Gamma_{i,j}(x, t, s)| \, ds \, dt \leqslant 40(M+N|x|)^2 B_2', \tag{A1.10}$$

for all x, t_0, $T \geqslant 0$. But obviously,

$$\begin{aligned}&\frac{1}{T} \int_{t_0}^{t_0+T} \int_{t_0}^{t_0+T} \Gamma(x, t, s) \, ds \, dt - A(x) \\ &\quad = \frac{1}{T} \int_{t_0}^{t_0+T} \int_{-\infty}^{\infty} \Gamma(x, t, s) \, ds \, dt - A(x) \\ &\quad\quad - \left\{ \int_{t_0}^{t_0+T} \int_{-\infty}^{t_0} \Gamma(x, t, s) \, ds \, dt + \int_{t_0}^{t_0+T} \int_{t_0+T}^{\infty} \Gamma(x, t, s) \, ds \, dt \right\} \end{aligned} \tag{A1.11}$$

Putting together (A1.11), (A1.10), (A1.9), (A1.8), and (A1.6) gives, for all $T>0$,

$$\sup_{\substack{|x|\leqslant D\\ t_0\geqslant 0}}\left|\frac{1}{T}\int_{t_0}^{t_0+T}\int_{t_0}^{t_0+T}\Gamma_{i,j}(x,t,s)\,ds\,dt-A_{i,j}(x)\right|$$

$$\leqslant\theta\left\{1-\frac{\theta}{T}\left[\frac{T}{\theta}\right]\right\}\sup_{|x|\leqslant D}|A_{i,j}(x)|+\frac{80}{T}(M+ND)^2(\theta B_1'+B_2'). \tag{A1.12}$$

Since $\sup_{|x|\leqslant D}|A_{i,j}(x)|<\infty$ (by Lemma A7.1), condition (C6) follows. ∎

APPENDIX 2: Facts about the Prohorov Metric

If $M(S)$ denotes the set of all probability measures on the Borel σ-algebra of a metric space (S,ρ) then the Prohorov distance between two $P, Q\in M(S)$ is given by

$$\Pi_S(P,Q)\triangleq\inf\{\varepsilon>0;\,P(A)\leqslant Q(A^\varepsilon)+\varepsilon\text{ for all closed }A\subset S\}, \tag{A2.1}$$

where $A^\varepsilon\triangleq\{x\in S;\ \rho(x,A)<\varepsilon\}$. That $\Pi_S(\cdot,\cdot)$ actually is a metric is established by Strassen (see Dudley [9, Proposition 1] and Prohorov [21, Section 1.4]). Moreover, when (S,ρ) is separable the Prohorov distance metricizes the topology of weak convergence in $M(S)$ (Dudley [9, Section 2]). A very noteworthy property of the Prohorov distance is given by Lemma A2.1 which is a special case of the Strassen–Dudley theorem (Dudley [9, Theorem 1]). Lemma A2.1 is used after line (3.28) in the proof of Proposition 1.

LEMMA A2.1. *Let (S,ρ) be a separable metric space and P_1, P_2 be two probability measures defined on the Borel sets of (S,ρ). Suppose that there is some $\alpha>0$ such that $\Pi_S(P_1,P_2)<\alpha$. Then there exists a probability measure Q on the Borel sets of $S\times S$ with marginals P_1 and P_2 (on the Borel sets of S) such that $Q\{(x,y):\rho(x,y)>\alpha\}\leqslant\alpha$.*

The following lemma, used in Eqs. (3.27), (3.36), and (3.58) of Section 3, is an almost inverse to Lemma A2.1. It is well known and easy to prove.

LEMMA A2.2. *Let (S,ρ) be a separable metric space and suppose X, Y are (S,ρ)-valued random elements on probability space $(\Omega,\mathscr{F},P)$.* (a) *If, for some $\beta>0$, we have $P\{\omega:\rho(X,Y)\geqslant\beta\}\leqslant\beta$ then the Prohorov distance between the probability measures induced on the Borel sets of (S,ρ) by X and Y satisfies $\Pi_S(\mathscr{L}(X),\mathscr{L}(Y))\leqslant\beta$.* (b) *If for some $\beta>0$ and real $r\geqslant1$, we have $\|\rho(X,Y)\|_r\leqslant\beta$ then $\Pi_S(\mathscr{L}(X),\mathscr{L}(Y))\leqslant\beta^{r/(r+1)}$.*

The following lemma is introduced implicitly by Yurinskii [24] and can be proved using simple real analysis. It is used in the proof of Lemma A6.1 (see line (A6.2)).

LEMMA A2.3. *Suppose that k is some fixed positive integer and $\{W_k(\tau), 0 \leqslant \tau \leqslant 1\}$ and $\{\tilde{W}_k(\tau), 0 \leqslant \tau \leqslant 1\}$ are $\Re^d$-valued processes on probability spaces $(\Omega, \mathscr{F}, P)$ and $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$, respectively, having continuous "piecewise linear" sample-paths*

$$W_k(\tau) = W_k\left(\frac{[\tau k]}{k}\right) + (\tau k - [\tau k])\left\{W_k\left(\frac{[\tau k]+1}{k}\right) - W_k\left(\frac{[\tau k]}{k}\right)\right\}$$

for $0 \leqslant \tau \leqslant 1$ and, similarly, for $\tilde{W}_k$. Moreover, suppose $W_k(0) = 0$ and $\tilde{W}_k(0) = 0$. Then

$$\Pi(\mathscr{L}(W_k), \mathscr{L}(\tilde{W}_k)) \leqslant \Pi_\infty^{kd}(\mathscr{L}(\Xi_k), \mathscr{L}(\tilde{\Xi}_k)),$$

where

$$\Xi_k \triangleq \begin{bmatrix} W_k(1/k) \\ W_k(2/k) \\ \vdots \\ W_k(1) \end{bmatrix}, \qquad \tilde{\Xi}_k \triangleq \begin{bmatrix} \tilde{W}_k(1/k) \\ \tilde{W}_k(2/k) \\ \vdots \\ \tilde{W}_k(1) \end{bmatrix} \tag{A2.2}$$

and $\Pi(\cdot, \cdot)$, $\Pi_\infty^{kd}(\cdot, \cdot)$ are the Prohorov metrics for probability measures on the Borel sets of $(C[0, 1], \|\cdot\|_C)$ and $(\Re^{kd}, |\cdot|)$, respectively (recall that $|x| \triangleq \max_{1 \leqslant i \leqslant kd} |x_i|$ for $x = (x_1 x_2 \cdots x_{kd})$ in $\Re^{kd}$).

LEMMA A2.4. *Let X and $\tilde{X}$ be k-dimensional random vectors on probability spaces $(\Omega, \mathscr{F}, P)$ and $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$, respectively. Then*

$$\Pi_\infty^k(\mathscr{L}(X), \mathscr{L}(\tilde{X})) \leqslant \Pi_2^k(\mathscr{L}(X), \mathscr{L}(\tilde{X})),$$

where $\Pi_\infty^k(\cdot, \cdot)$ and $\Pi_2^k(\cdot, \cdot)$ are the Prohorov metrics for probability measures on the Borel sets of $(\Re^k, |\cdot|)$ and $(\Re^k, |\cdot|_2)$ (recall that $|x|_2 \triangleq (\sum_{i=1}^{kd} x_i^2)^{1/2}$ for $x = (x_1 x_2 \cdots x_{kd})$ in $\Re^{kd}$).

APPENDIX 3: USEFUL RESULTS

In this appendix we collect for easy reference four crucial theorems on which this note is based. The following maximal inequality is used many times in this note. It is a simple consequence of a maximal inequality of Longnecker and Serfling [19, Theorem 1] along with discretization and passage to a continuous limit (see Proposition A1 in [14]).

THEOREM A3.1. *Let* $0 \leqslant T < U < \infty$ *be constants and suppose that* $\{Q_t, T \leqslant t \leqslant U\}$ *is a process assuming values in* $\Re^d$ *(with norm* $|\cdot|$ *specified in* (C0), *Section* 2) *such that*

(i) $t \to Q_t(\omega)$ *is continuous on* $[T, U]$ *for almost all* ω, *and*

(ii) *there exist constants* $\gamma > 1$, $v > 0$ *such that*

$$E\,|Q_u - Q_t|^v \leqslant [h(t, u)]^\gamma \qquad \text{for all} \quad T \leqslant t \leqslant u \leqslant U,$$

where $h(t, u)$ *is a non-negative function satisfying*

(iii) $h(t, u) + h(u, v) \leqslant h(t, v)$ *for all* $T \leqslant t < u < v \leqslant U$.

Then there exists a constant $\tilde{A}_{v,\gamma}$ *depending only on* v *and* γ *such that*

$$E[\max_{T \leqslant t \leqslant u \leqslant U} |Q_u - Q_t|^v] \leqslant \tilde{A}_{v,\gamma}[h(T, U)]^\gamma.$$

The following result is used frequently throughout this paper and, in particular, for the development of the bounds of Appendix 4.

LEMMA A3.2 (Davydov [6, Lemma 7]; Deo [8, Lemma 1]). *Let* ξ *and* η *be* $\mathscr{G}$*-measurable and* $\mathscr{H}$*-measurable real-valued random variables respectively. Let* $r, s, t > 1$ *be constants such that* $r^{-1} + s^{-1} + t^{-1} = 1$ *and* $\|\xi\|_s < \infty$ *and* $\|\eta\|_t < \infty$. *Then*

$$|E\xi\eta - E\xi E\eta| \leqslant 10(\alpha(\mathscr{G}, \mathscr{H}))^{1/r}\, \|\xi\|_s\, \|\eta\|_t,$$

where

$$\alpha(\mathscr{G}, \mathscr{H}) \triangleq \sup_{\substack{A \in \mathscr{G} \\ B \in \mathscr{H}}} |P(A \cap B) - P(A)P(B)|.$$

The following proposition is the key tool for establishing a bound on the second term on the right-hand side of (A6.31) in the proof of Lemma A6.2.

PROPOSITION A3.3. *Let* $X_1, ..., X_n$ *be zero mean,* $\Re^m$*-valued random vectors with* $\hat{M} \triangleq \sup_{1 \leqslant i \leqslant n} E\,|X_i|_2^3 < \infty$. *Let* $\mathscr{M}_a^b$ *denote the* σ*-field generated by* $X_a, ..., X_b$ *and define*

$$\beta \triangleq \sup_{0 < k < n} \sup_{\substack{A \in \mathscr{M}_1^k \\ B \in \mathscr{M}_{k+1}^n}} |P(A \cap B) - P(A)P(B)|.$$

Then there exists an absolute constant $\tilde{c} > 0$ *such that*

$$\Pi_2^m\left(\mathscr{L}(n^{-1/2}(X_1 + \cdots + X_n)), \mathscr{N}\left(0, n^{-1} \sum_{i=1}^{n} \operatorname{cov} X_i\right)\right)$$
$$\leqslant \tilde{c}(\beta^{2/3} n^{1/2} \zeta^{-1} m^{-1/2} \hat{M}^{1/3} + m^{3/2} \beta^{1/3} \hat{M}^{2/3} \zeta^{-2} + n^{-1/2} \hat{M} \zeta^{-3} m^{-1/2})$$
$$+ 4\zeta + 4\zeta \left(\log \frac{1}{\zeta}\right)^{1/2} m^{1/2} + 4\zeta m^{1/2}$$

for all $\zeta \in (0, 1)$, *where* $\Pi_2^m(\cdot, \cdot)$ *is the Prohorov metric for probability measures on the Borel sets of* $(\mathfrak{R}^m, |\cdot|_2)$.

Proof. This result is Eq. (6.2) on page 418 of Dehling [7]. It is developed in Lemma 2.3, Lemma 2.4, and Proposition 6.1 of [7].

The following theorem is used for establishing a bound on the third term on the right-hand side of (A6.31) in Lemma A6.2.

THEOREM A3.4. *Let* T, S *be* k *by* k *symmetric, positive semi-definite matrices. Then there exists an absolute constant* $\hat{c} > 0$ *such that*

$$\Pi_2^k(\mathscr{N}(0, T), \mathscr{N}(0, S)) \leqslant \hat{c}\, |||T - S|||_1^{1/3}\, k^{1/6}(1 + |\log(|||T - S|||_1^{-1}\, k)|^{1/2}),$$

where $|||A|||_1$ *denotes the trace class norm of matrix* A.

Proof. This theorem is due to Dehling [7, Theorem 7]. The statement of the theorem is on page 400 and the proof is on page 406 of [7].

Remark A3.1. For a k by k matrix A, $|||A|||_1$ is defined to be the sum of the singular values of A (see pp. 170–173 of Weidmann [23] for a general treatment). When A is symmetric, clearly $|||A|||_1 \triangleq \sum_{i=1}^{k} |\lambda_i|$, where λ_i are the eigenvalues of A. If $|A|_2 \triangleq (\sum_{i,j=1}^{k} a_{ij}^2)^{1/2}$ denotes the Frobenius norm of a symmetric matrix A with eigenvalues λ_i then clearly $k^2\, |A|_2^2 = k^2 \sum_{i=1}^{k} |\lambda_i|^2 \geqslant \{|||A|||_1\}^2$. This will be used in the proof of Lemma A6.2 (see line (A6.55).

APPENDIX 4: MOMENT BOUNDS

In this appendix we give three moment bounds for zero-mean strong mixing processes, stated as parts (A), (B), and (C) of Lemma A4. The first two bounds are essentially due to Khas'minskii [17] while the third is developed here; unfortunately its proof, although simple in concept, is rather long and tedious. Lemma A4(A) is used on numerous occasions, Lemma A4(B) is needed for line (A7.2), and Lemma A4(C) is used for line (A5.23).

LEMMA A4. *For some positive integer k, let $\Phi_1(t), \Phi_2(t), ..., \Phi_{2k}(t)$, $t \geqslant 0$, be zero mean stochastic processes on some probability space $(\Omega, \mathscr{F}, P)$. Suppose each $\Phi_i(t)$ is $\mathscr{F}^t_t$-measurable, where $\{\mathscr{F}^t_s, 0 \leqslant s \leqslant t \leqslant \infty\}$ satisfies condition* (C2) *of Section* 2 *with some function* $\alpha(\cdot)$.

(A) *Suppose for some* $\delta > 0$ *that*

(i) $M \triangleq \sup_{t>0,\, i=1,2,...,2k} \|\Phi_i(t)\|_{(2+\delta)k} < \infty$

(ii) $R'_n \triangleq \int_0^\infty \tau^{n-1}[\alpha(\tau)]^{\delta/(2+\delta)}\, d\tau < \infty$, $n = 1, 2, ..., k$.

Then there exists a constant c_{2k}, depending only on k, M, $R'_1, ..., R'_k$, such that for all $0 \leqslant t \leqslant u$,

$$\int_t^u \cdots \int_t^u |E\{\Phi_1(s_1) \cdots \Phi_{2k}(s_{2k})\}|\, ds_1 \cdots ds_{2k} \leqslant c_{2k}(u-t)^k.$$

(B) *Suppose there is some number L such that*

(i) $|\Phi_i(t, \omega)| \leqslant L$ *for all* $t \geqslant 0$, $i = 1, 2, ..., 2k$ *a.a.* $\omega \in \Omega$,

(ii) $R_n \triangleq \int_0^\infty \tau^{n-1}[\alpha(\tau)]\, d\tau < \infty$, $n = 1, 2, ..., k$.

Then there exists a constant c_{2k}, depending only on k, $R_1, ..., R_k$, such that for all $0 \leqslant t \leqslant u$,

$$\int_t^u \cdots \int_t^u |\{E\Phi_1(s_1) \cdots \Phi_{2k}(s_{2k})\}|\, ds_1 \cdots ds_{2k} \leqslant c_{2k} L^{2k}(u-t)^k.$$

(C) *Suppose for some* $\delta > 0$ *that*

(i) $M \triangleq \sup_{t \geqslant 0,\, i=1,...,8} \|\Phi_i(t)\|_{8+4\delta} < \infty$

(ii) $R'_n \triangleq \int_0^\infty \tau^{n-1}[\alpha(\tau)]^{\delta/(2+\delta)}\, d\tau < \infty$, $n = 1, 2, 3, 4$.

Then there exists a constant c_8, depending only on M, $R'_1, ..., R'_4$, such that for all $0 \leqslant t \leqslant u$,

$$\int_t^u \cdots \int_t^u \left\{\int_0^t \cdots \int_0^t |E\{\Phi_1(s_1) \cdots \Phi_4(s_4)\Phi_5(v_1) \cdots \Phi_8(v_4)\}|\, dv_1 \cdots dv_4\right\}$$
$$\times\, ds_1 \cdots ds_4 \leqslant c_8 t^2(u-t)^2.$$

Remark A4.1. Comparing cases (A) and (B), one sees that the strong absolute bound in (B)(i) results in a more structured bound on the right of the $2k$-fold iterated integral. This structure will be used in the proof of Lemma A7.1.

Remark A4.2. Lemma A4(A) is a minor extension of Lemma 2.1 of Khas'minskii [17] and is proved in Lemma A2.1 of Heunis and Kouritzin [14]. (Khas'minskii postulates a somewhat stronger moment bound than that in (A)(i) to obtain the conclusion of Lemma A4(A)). Lemma A4(B) follows inter alia from the proof of Lemma 2.1 in Khas'minskii (see the line following Eq. (2.10) on page 215 of [17]).

Proof of Lemma A4(C). (For ease of notation we write $\tilde{\Phi}_k$ for Φ_{4+k}, $k=1, ..., 4$.) The proof is divided into two steps.

(I) We first show that there exists some c' depending only on M, R_1', R_2', R_3' such that

$$\int_t^u \cdots \int_t^u \int_0^t \int_0^t |E\{\Phi_1(s_1) \cdots \Phi_4(s_4) \tilde{\Phi}_1(v_1) \tilde{\Phi}_2(v_2)\}| \, dv_2 \, dv_1 \, ds_4 \cdots ds_1$$
$$\leqslant c'(u-t)^2 t. \tag{A4.1}$$

Fix numbers $\{s_1, s_2, s_3, s_4\} \in [t, u]$ and $\{v_1, v_2\} \in [0, t]$ and let $\{j_k, k=1, 2, 3, 4\}$ and $\{i_k, k=1, 2\}$ denote the subscripts of $\{s_1, ..., s_4\}$ respectively $\{v_1, v_2\}$ such that:

$$0 \leqslant v_{i_2} \leqslant v_{i_1} \leqslant t \leqslant s_{j_1} \leqslant s_{j_2} \leqslant s_{j_3} \leqslant s_{j_4} \leqslant u \tag{A4.2}$$

Then, by the Davydov bound (Lemma A3.2 with $r \triangleq (\delta+2)/\delta$, $s \triangleq (6+3\delta)/4$, and $t \triangleq (6+3\delta)/2$), repeated use of Hölder's inequality and hypothesis C(i):

$$\begin{aligned}
&|E\{\Phi_{j_1}(s_{j_1}) \cdots \Phi_{j_4}(s_{j_4}) \tilde{\Phi}_{i_1}(v_{i_1}) \tilde{\Phi}_{i_2}(v_{i_2})\} \\
&\quad - E\{\Phi_{j_1}(s_{j_1}) \cdots \Phi_{j_4}(s_{j_4})\} E\{\tilde{\Phi}_{i_1}(v_{i_1}) \tilde{\Phi}_{i_2}(v_{i_2})\}| \\
&\leqslant 10\{\alpha(s_{j_1} - v_{i_1})\}^{\delta/(\delta+2)} \|\Phi_{j_1}(s_{j_1}) \cdots \Phi_{j_4}(s_{j_4})\|_{(6+3\delta)/4} \\
&\quad \times \|\tilde{\Phi}_{i_1}(v_{i_1}) \tilde{\Phi}_{i_2}(v_{i_2})\|_{(6+3\delta)/2} \\
&\leqslant 10\{\alpha(s_{j_1} - v_{i_1})\}^{\delta/(\delta+2)} \|\Phi_{j_1}(s_{j_1})\|_{6+3\delta} \cdots \|\Phi_{j_4}(s_{j_4})\|_{6+3\delta} \\
&\quad \cdots \|\tilde{\Phi}_{i_1}(v_{i_1})\|_{6+3\delta} \|\tilde{\Phi}_{i_2}(v_{i_2})\|_{6+3\delta} \\
&\leqslant 20M^6\{\alpha(s_{j_1} - v_{i_1})\}^{\delta/(\delta+2)}.
\end{aligned} \tag{A4.3}$$

Also, by (A4.2), Lemma A3.2, the zero mean property of Φ_{j_4}, repeated use of Hölder's inequality, and hypothesis C(i):

$$\begin{aligned}
&|E\{\Phi_{j_1}(s_{j_1}) \cdots \Phi_{j_4}(s_{j_4}) \tilde{\Phi}_{i_1}(v_{i_1}) \tilde{\Phi}_{i_2}(v_{i_2})\} \\
&\quad - E\{\Phi_{j_1}(s_{j_1}) \cdots \Phi_{j_4}(s_{j_4})\} E\{\tilde{\Phi}_{i_1}(v_{i_1}) \tilde{\Phi}_{i_2}(v_{i_2})\}| \\
&\leqslant |E\{\Phi_{j_1}(s_{j_1}) \cdots \Phi_{j_4}(s_{j_4}) \tilde{\Phi}_{i_1}(v_{i_1}) \tilde{\Phi}_{i_2}(v_{i_2})\} \\
&\quad + |E\{\Phi_{j_1}(s_{j_1}) \cdots \Phi_{j_4}(s_{j_4})\}| \, |E\{\tilde{\Phi}_{i_1}(v_{i_1}) \tilde{\Phi}_{i_2}(v_{i_2})\}|
\end{aligned}$$

$$\leqslant 10\{\alpha(s_{j_4}-s_{j_3})\}^{\delta/(\delta+2)}\{\|\Phi_{j_4}(s_{j_4})\|_{6+3\delta}$$
$$\times\|\Phi_{j_1}(s_{j_1})\Phi_{j_2}(s_{j_2})\Phi_{j_3}(s_{j_3})\tilde{\Phi}_{i_1}(v_{i_1})\tilde{\Phi}_{i_2}(v_{i_2})\|_{(6+3\delta)/5}$$
$$+\|\Phi_{j_4}(s_{j_4})\|_{4+2\delta}\|\Phi_{j_1}(s_{j_1})\Phi_{j_2}(s_{j_2})\Phi_{j_3}(s_{j_3})\|_{(4+2\delta)/3}\cdot|E\tilde{\Phi}_{i_1}(v_{i_1})\tilde{\Phi}_{i_2}(v_{i_2})|\}$$
$$\leqslant 10[\alpha(s_{j_4}-s_{j_3})]^{\delta/(\delta+2)}\{M\cdot M^5+M\cdot M^3\cdot M^2\}$$
$$=20M^6\cdot\{\alpha(s_{j_4}-s_{j_3})\}^{\delta/(\delta+2)}. \tag{A4.4}$$

Similarly, by (A4.2), Lemma A3.2, the zero mean property of $\tilde{\Phi}_{i_2}$, Hölder's inequality, and hypothesis C(i):

$$|E\{\Phi_{j_1}(s_{j_1})\cdots\Phi_{j_4}(s_{j_4})\tilde{\Phi}_{i_1}(v_{i_1})\tilde{\Phi}_{i_2}(v_{i_2})\}$$
$$-E\{\Phi_{j_1}(s_{j_1})\cdots\Phi_{j_4}(s_{j_4})\}E\{\tilde{\Phi}_{i_1}(v_{i_1})\tilde{\Phi}_{i_2}(v_{i_2})\}|$$
$$\leqslant 20M^6\{\alpha(v_{i_1}-v_{i_2})\}^{\delta/(\delta+2)}. \tag{A4.5}$$

Since (A4.3), (A4.4), and (A4.5) must hold simultaneously and we can remove i and j from the subscripts in the quantities on the left of (A4.3), (A4.4), and (A4.5) without changing their values, we have by the monotonicity of $\alpha(\cdot)$:

$$|E\{\Phi_1(s_1)\cdots\Phi_4(s_4)\tilde{\Phi}_1(v_1)\tilde{\Phi}_2(v_2)\}$$
$$-E\{\Phi_1(s_1)\cdots\Phi_4(s_4)\}E\{\tilde{\Phi}_1(v_1)\tilde{\Phi}_2(v_2)\}|$$
$$\leqslant 20M^6[\alpha(\max\{(s_{j_4}-s_{j_3}),(s_{j_1}-v_{i_1}),(v_{i_1}-v_{i_2})\})]^{\delta/(\delta+2)}$$
$$\leqslant 20M^6\left[\alpha\left(\frac{(s_{j_4}-s_{j_3})+(s_{j_1}-v_{i_1})+(v_{i_1}-v_{i_2})}{3}\right)\right]^{\delta/(\delta+2)}$$
$$\triangleq \beta(s_1,s_2,s_3,s_4,v_1,v_2). \tag{A4.6}$$

Now the indices j_k, i_k defined by (A4.2) are functions of $\{s_1,...,s_4\}$ and $\{v_1,v_2\}$ such that $\beta(s_1,s_2,s_3,s_4,v_1,v_2)$ is unchanged by any of the 4! permutations of $\{s_1,...,s_4\}$ and 2! permutations of $\{v_1,v_2\}$. Thus, by part (A) of Lemma A4 there exist constants $c_4>0$, depending only on M, R_1', R_2', and $c_2>0$, depending only on M, R_1', such that

$$\int_t^u\cdots\int_t^u\int_0^t\int_0^t|E\{\Phi(s_1)\cdots\Phi(s_4)\tilde{\Phi}(v_1)\tilde{\Phi}(v_2)\}|\,dv_2\,dv_1\,ds_4\cdots ds_1$$
$$\leqslant\int_t^u\int_t^u\int_t^u\int_t^u\int_0^t\int_0^t|E\{\Phi_1(s_1)\cdots\Phi_4(s_4)\tilde{\Phi}_1(v_1)\tilde{\Phi}_2(v_2)\}$$
$$-E\{\Phi_1(s_1)\cdots\Phi_4(s_4)\}E\{\tilde{\Phi}_1(v_1)\tilde{\Phi}_2(v_2)\}|\,dv_2\,dv_1\,ds_4\cdots ds_1$$
$$+\int_t^u\int_t^u\int_t^u\int_t^u\int_0^t\int_0^t|E\{\Phi_1(s_1)\cdots\Phi_4(s_4)\}E\{\tilde{\Phi}_1(v_1)\tilde{\Phi}_2(v_2)\}|$$
$$\times dv_2\,dv_1\,ds_4\cdots ds_1$$

$$\leqslant \int_t^u \int_t^u \int_t^u \int_t^u \int_0^t \int_0^t \beta(s_1, s_2, s_3, s_4, v_1, v_2)\, dv_2\, dv_1\, ds_4 \cdots ds_1$$

$$+ \int_t^u \int_t^u \int_t^u \int_t^u |E\{\Phi_1(s_1) \cdots \Phi_4(s_4)\}|$$

$$\times ds_4 \cdots ds_1 \int_0^t \int_0^t |E\{\tilde{\Phi}_1(v_1)\tilde{\Phi}_2(v_2)\}|\, dv_2\, dv_1$$

$$\leqslant 4!\, 2 \int_t^u \int_{s_1}^u \int_{s_2}^u \int_{s_3}^u \int_0^t \int_0^{v_1} 20M^6$$

$$\times \left[\alpha \left(\frac{(s_4 - s_3) + (s_1 - v_1) + (v_1 - v_2)}{3} \right) \right]^{\delta/(\delta+2)}$$

$$\times dv_2\, dv_1\, ds_4 \cdots ds_1 + c_4 c_2 (u-t)^2 t. \tag{A4.7}$$

We now bound the first term on the far right-hand side of (A4.7). Define $x_1 \triangleq v_1 - v_2$, $x_2 \triangleq s_1 - v_1$, $x_3 \triangleq s_4 - s_3$, $x_4 \triangleq s_3 - s_2$, $x_5 \triangleq s_2 - s_1$, $x_6 \triangleq v_2$, and note that the corresponding Jacobian is 1. Also, we note that $0 \leqslant x_1$, $x_6 \leqslant t$, $0 \leqslant x_3$, $x_4, x_5 \leqslant u-t$, and $0 \leqslant x_2 \leqslant u$. Substituting these variables into (A4.7) we have by a change of variables and the non-negativity of $\alpha(\cdot)$:

$$\int_t^u \cdots \int_t^u \int_0^t \int_0^t |E\{\Phi_1(s_1) \cdots \Phi_4(s_4) \tilde{\Phi}_1(v_1) \tilde{\Phi}_2(v_2)\}|\, dv_2\, dv_1\, ds_4\, ds_3\, ds_2\, ds_1$$

$$\leqslant 4! \cdot 40M^6 \int_0^t \int_0^{u-t} \int_0^{u-t} \int_0^{u-t} \int_0^u \int_0^t \left[\alpha \left(\frac{x_3 + x_2 + x_1}{3} \right) \right]^{\delta/(\delta+1)}$$

$$\times dx_1 \cdots dx_6 + c_2 c_4 (u-t)^2 t$$

$$\leqslant 960 M^6 (u-t)^2 t \int_0^\infty \int_0^\infty \int_0^\infty \left[\alpha \left(\frac{x_3 + x_2 + x_1}{3} \right) \right]^{\delta/(\delta+2)}$$

$$\times dx_1\, dx_2\, dx_3 + c_2 c_4 (u-t)^2 t$$

$$\leqslant 960 M^6 (u-t)^2 t \frac{27}{2} R_3' + c_2 c_4 (u-t)^2 t, \tag{A4.8}$$

where we have used the easily verified fact that

$$\int_0^\infty \cdots \int_0^\infty [\alpha(t_1 + \cdots + t_{r+1})]^{\delta/(\delta+2)}\, dt_1 \cdots dt_{r+1}$$

$$= \frac{1}{r!} \int_0^\infty u^r [\alpha(u)]^{\delta/(\delta+2)}\, du \tag{A4.9}$$

for $r=0, 1, 2, 3$, which follows from condition (C)(ii) of the lemma; (A4.9) gives (I).

(II) We now show assertion (C) of the lemma using part (I) just established. Fix numbers $\{s_1, s_2, s_3, s_4\} \in [t, u]$ and $\{v_1, v_2, v_3, v_4\} \in [0, t]$ and let $\{j_k, k=1, 2, 3, 4\}$ and $\{i_k, k=1, 2, 3, 4\}$ denote the subscripts of $\{s_1, ..., s_4\}$ respectively $\{v_1, ..., v_4\}$ such that

$$0 \leqslant v_{i_4} \leqslant v_{i_3} \leqslant v_{i_2} \leqslant v_{i_1} \leqslant t \leqslant s_{j_1} \leqslant s_{j_2} \leqslant s_{j_3} \leqslant s_{j_4} \leqslant u. \tag{A4.10}$$

Now as in part (I) (A4.6), using Lemma A3.2, the zero mean property of Φ_{j_4}, $\tilde{\Phi}_{i_4}$, repeated use of Hölder's inequality, and hypothesis C(i), we find:

$$\begin{aligned}|E\{&\Phi_1(s_1)\cdots\Phi_4(s_4)\tilde{\Phi}_1(v_1)\cdots\tilde{\Phi}_4(v_4)\}\\&-E\{\Phi_{j_1}(s_{j_1})\cdots\Phi_{j_4}(s_{j_4})\}E\{\tilde{\Phi}_{i_1}(v_{i_1})\cdots\tilde{\Phi}_{i_4}(v_{i_4})\}|\\&\leqslant 20\cdot M^8[\alpha(\max\{(s_{j_4}-s_{j_3}), (s_{j_1}-v_{i_1}), (v_{i_3}-v_{i_4})\}]^{\delta/(\delta+2)}\end{aligned} \tag{A4.11}$$

and

$$\begin{aligned}|E\{&\Phi_1(s_1)\cdots\Phi_4(s_4)\tilde{\Phi}_1(v_1)\cdots\tilde{\Phi}_4(v_4)\}\\&-E\{\Phi_{j_1}(s_{j_1})\cdots\Phi_{j_4}(s_{j_4})\tilde{\Phi}_{i_1}(v_{i_1})\tilde{\Phi}_{i_2}(v_{i_2})\}E\{\tilde{\Phi}_{i_3}(v_{i_3})\tilde{\Phi}_{i_4}(v_{i_4})\}|\\&\leqslant 20\cdot M^8[\alpha(\max\{(s_{j_4}-s_{j_3}), (v_{i_2}-v_{i_3}), (v_{i_3}-v_{i_4})\})]^{\delta/(\delta+2)}.\end{aligned} \tag{A4.12}$$

Now put

$$\begin{aligned}A &\triangleq E\{\Phi_1(s_1)\cdots\Phi_4(s_4)\tilde{\Phi}_1(v_1)\cdots\tilde{\Phi}_4(v_4)\}\\B &\triangleq E\{\Phi_{j_1}(s_{j_1})\cdots\Phi_{j_4}(s_{j_4})\}E\{\tilde{\Phi}_{i_1}(v_{i_1})\cdots\tilde{\Phi}_{i_4}(v_{i_4})\}\\C &\triangleq E\{\Phi_{j_1}(s_{j_1})\cdots\Phi_{j_4}(s_{j_4})\tilde{\Phi}_{i_1}(v_{i_1})\tilde{\Phi}_{i_2}(v_{i_2})\}E\{\tilde{\Phi}_{i_3}(v_{i_3})\tilde{\Phi}_{i_4}(v_{i_4})\}.\end{aligned}$$

Then from (A4.11), (A4.12), and the fact that $\alpha(\cdot)$ is non-increasing, we obtain

$$\begin{aligned}|A| &\leqslant \min\{|A-B|, |A-C|\}+|B|+|C|\\&\leqslant 20M^8[\alpha(\max\{(s_{j_4}-s_{j_3}), (s_{j_1}-v_{i_1}), (v_{i_2}-v_{i_3}), (v_{i_3}-v_{i_4})\})]^{\delta/(\delta+2)}\\&\quad+|B|+|C|\\&\leqslant 20M^8\left[\alpha\left(\frac{(s_{j_4}-s_{j_3})+(s_{j_1}-v_{i_1})+(v_{i_2}-v_{i_3})+(v_{i_3}-v_{i_4})}{4}\right)\right]^{\delta/(\delta+2)}\\&\quad+|B|+|C|.\end{aligned} \tag{A4.13}$$

Now as in (A4.7), we have by (A4.13), Lemma A4(A), and part (I) that there exists constant $K(=4!\,4!\,20M^6)>0$ such that

$$\int_t^u \cdots \int_t^u \int_0^t \cdots \int_0^t |E\{\Phi_1(s_1)\cdots\Phi_4(s_4)\tilde{\Phi}_1(v_1)\cdots\tilde{\Phi}_4(v_4)\}|$$

$$\times\, dv_4\, dv_3\, dv_2\, dv_1\, ds_4\, ds_3\, ds_2\, ds_1$$

$$\leqslant K\int_t^u \int_{s_1}^u \int_{s_2}^u \int_{s_3}^u \int_0^t \int_0^{v_1} \int_0^{v_2} \int_0^{v_3}$$

$$\left[\alpha\left(\frac{s_4-s_3+s_1-v_1+v_2-v_4}{4}\right)\right]^{\delta/(\delta+2)} dv_4\cdots dv_1\, ds_4\cdots ds_1$$

$$+\int_t^u \cdots \int_t^u |E\{\Phi_1(s_1)\cdots\Phi_4(s_4)\}|$$

$$\times\, ds_4\, ds_3\, ds_2\, ds_1 \int_0^t \cdots \int_0^t |E\{\tilde{\Phi}_1(v_1)\cdots\tilde{\Phi}_4(v_4)\}|\, dv_4\, dv_3\, dv_2\, dv_1$$

$$+\sum_{\substack{i_1,i_2,i_3,i_4\in\{1,\ldots,4\}\\ i_1,i_2,i_3,i_4 \text{ distinct}}} \int_t^u \cdots \int_t^u \int_0^t \int_0^{v_1} \int_0^{v_2} \int_0^{v_3}$$

$$|E\{\Phi_1(s_1)\cdots\Phi_4(s_4)\tilde{\Phi}_{i_1}(v_1)\tilde{\Phi}_{i_2}(v_2)\}|$$

$$\times\, |E\{\tilde{\Phi}_{i_3}(v_3)\tilde{\Phi}_{i_4}(v_4)\}|\, dv_4\, dv_3\, dv_2\, dv_1\, ds_4\, ds_3\, ds_2\, ds_1$$

$$\leqslant K\int_t^u \int_{s_1}^u \int_{s_2}^u \int_{s_3}^u \int_0^t \int_0^{v_1} \int_0^{v_2} \int_0^{v_3}$$

$$\left[\alpha\left(\frac{s_4-s_3+s_1-v_1+v_2-v_4}{4}\right)\right]^{\delta/(\delta+2)} dv_4\cdots dv_1\, ds_4\cdots ds_1$$

$$+c_4^2(u-t)^2t^2+4!\,c'c_2(ut)^2t^2, \tag{A4.14}$$

where c' is the constant in (A4.1). Now let $x_1 \triangleq v_3-v_4$, $x_2 \triangleq v_2-v_3$, $x_3 \triangleq s_1-v_1$, $x_4 \triangleq s_4-s_3$, $x_5 \triangleq s_3-s_2$, $x_6 \triangleq s_2-s_1$, $x_7 \triangleq v_1-v_2$, $x_8 \triangleq v_4$ and note that the corresponding Jacobian is 1 and that $0\leqslant x_1, x_2, x_7, x_8 \leqslant t$, $0\leqslant x_4, x_5, x_6 \leqslant u-t$, and $0\leqslant x_3\leqslant u$. Then from (A4.14), the monotonicity of $\alpha(\cdot)$, and (A4.9) we have

$$\int_t^u \cdots \int_t^u \int_0^t \cdots \int_0^t |E\{\Phi_1(s_1)\cdots\Phi_4(s_4)\tilde{\Phi}_1(v_1)\cdots\tilde{\Phi}_4(v_4)\}|\, dv_4\cdots dv_1 ds_4\cdots ds_1$$

$$\leqslant K(u-t)^2t^2\int_0^{u-t}\int_0^u\int_0^t\int_0^t\left[\alpha\left(\sum_{i=1}^4 \frac{x_i}{4}\right)\right]^{\delta/(\delta+2)}$$

$$\times\, dx_1\, dx_2\, dx_3\, dx_4+(c_4^2+4!\,c'c_2)(u-t)^2t^2$$

$$\leqslant 4^4 K(u-t)^2 t^2 \int_0^\infty \cdots \int_0^\infty \left[\alpha\left(\sum_{i=1}^4 t_i\right)\right]^{\delta/(\delta+2)}$$

$$\times dt_1\, dt_2\, dt_3\, dt_4 + (c_4^2 + 4!\,c'c_2)(u-t)^2 t^2$$

$$\leqslant \left(\frac{K}{6} 4^4 R_4' + c_4^2 + 4!\,c'c_2\right)(u-t)^2 t^2. \quad \blacksquare \tag{A4.15}$$

APPENDIX 5: Maximal Bounds

This appendix contains maximal moment bounds used in the proof of Proposition 1. Lemma A5.1 is used in Eq. (3.13) of Section 3.

Lemma A5.1. *Under conditions* (C0)–(C5) *and* (C7) *of Section* 2 *there exists some* $c_3 > 0$ *depending only on* M, N, D, d, *and* B_1', B_2' *such that*

$$E[\max_{0 \leqslant \tau \leqslant 1} |Y^\varepsilon(\tau)|^2] \leqslant c_3 \qquad \textit{for all} \quad 0 < \varepsilon \leqslant 1,$$

where $Y^\varepsilon(\cdot)$ *is defined in Eq.* (2.13).

Proof. Fix an $\varepsilon \in (0, 1]$ and an $\omega \in \Omega$. By (2.3), (2.7), (2.1), (2.11), and (2.9):

$$|X^\varepsilon(\tau) - x^0(\tau)| \leqslant N \int_0^\tau |X^\varepsilon(s) - x^0(s)|\, ds$$

$$+ \max_{0 \leqslant \tau \leqslant 1} \left| \int_0^\tau \tilde{F}(x^0(s), s/\varepsilon)\, ds \right| + c\varepsilon \tag{A5.1}$$

for all $0 \leqslant \tau \leqslant 1$ and so by Gronwall, (2.13), and a change of variables,

$$\max_{0 \leqslant \tau \leqslant 1} |Y^\varepsilon(\tau)| \leqslant \varepsilon^{1/2} e^N \cdot \max_{0 \leqslant t \leqslant \varepsilon^{-1}} \left| \int_0^t \tilde{F}(x^0(\varepsilon s), s)\, ds \right| + e^N c \varepsilon^{1/2} \tag{A5.2}$$

for all $\omega \in \Omega$. Therefore,

$$E[\max_{0 \leqslant \tau \leqslant 1} |Y^\varepsilon(\tau)|^2]$$

$$\leqslant 2e^{2N} \left\{ \varepsilon E\left[\max_{0 \leqslant t \leqslant \varepsilon^{-1}} \left| \int_0^t \tilde{F}(x^0(\varepsilon s), s)\, ds \right|^2 \right] + c^2 \right\}. \tag{A5.3}$$

Now by Lemma A7.2(ii) for all $0 \leqslant s \leqslant \varepsilon^{-1}$, $i = 1, ..., d$:

$$\|\tilde{F}_i(x^0(\varepsilon s), s)\|_{4+2\delta} \leqslant 2M + 2ND. \tag{A5.4}$$

Moreover, $E\tilde{F}_i(x^0(\varepsilon s), s) = 0$ for all $0 \leqslant s \leqslant \varepsilon^{-1}$ and $i = 1, ..., d$, so by (2.5) and Lemma A4(A) of Appendix 4 (applying it for each of the d-components and taking $k \triangleq 4$) there exists some $c_1 > 0$ (depending only on M, N, D, d, B_1', B_2' of Section 2) such that

$$E\left|\int_t^u \tilde{F}(x^0(\varepsilon s), s)\, ds\right|^4 \leqslant E\left(\sum_{i=1}^{d}\left|\int_t^u \tilde{F}_i(x^0(\varepsilon s), s)\, ds\right|^4\right) \leqslant c_1(u-t)^2 \tag{A5.5}$$

for all $0 \leqslant t \leqslant u \leqslant \varepsilon^{-1}$. Defining $h(t, u) \triangleq c_1^{1/2}(u-t)$, by Theorem A3.1 (with $v \triangleq 4$, $\gamma \triangleq 2$, $Q_t \triangleq \int_0^t \tilde{F}(x^0(\varepsilon s), s)\, ds$, $T \triangleq 0$, and $U \triangleq \varepsilon^{-1}$) there is some constant $c_2 > 0$ (depending only on M, N, D, d, and B_1', B_2') such that

$$E\left[\max_{0 \leqslant t \leqslant \varepsilon^{-1}}\left|\int_0^t \tilde{F}(x^0(\varepsilon s), s)\, ds\right|^4\right] \leqslant c_2 \varepsilon^{-2}. \tag{A5.6}$$

The lemma follows with $c_3 = 2e^{2N}(c_2^{1/2} + c^2)$ from (A5.3), (A5.6), and Hölder's inequality. ∎

The bounds developed in Lemmas A5.2 and A5.3 are used in Eq. (3.25), Section 3.

LEMMA A5.2. *Let $A^\varepsilon(v)$ be defined as in* (3.21) *for $0 \leqslant v \leqslant 1$. Then under conditions* (C0)–(C5) *of Section* 2 *there exists a constant $c_4 > 0$, depending only on constants M, N, D, d, and $B_1', ..., B_4'$ of Section* 2 *such that for all $\varepsilon > 0$,*

$$E\,\|A^\varepsilon\|_C^4 \leqslant c_4 \cdot \varepsilon^2.$$

Proof. Fix $\varepsilon > 0$ and put $\Phi^\varepsilon(s) \triangleq \tilde{F}(x^0(\varepsilon s), s)$ and $\tilde{\Phi}^\varepsilon(s) \triangleq [(\partial F/\partial x)(x^0((\varepsilon s), s) - E((\partial F/\partial x)(x^0(\varepsilon s), s))]$ for $0 \leqslant s \leqslant \varepsilon^{-1}$. By a change of variables, (3.1), and then a further change of variables,

$$\begin{aligned} A^\varepsilon(v) &= \int_0^v \left[\frac{\partial F}{\partial x}(x^0(s), s/\varepsilon) - E\left(\frac{\partial F}{\partial x}(x^0(s), s/\varepsilon)\right)\right] W_1^\varepsilon(s)\, ds \\ &= \varepsilon^{1/2} \int_0^{v/\varepsilon} \tilde{\Phi}^\varepsilon(s) \int_0^{\varepsilon s} \tilde{F}(x^0(u), u/\varepsilon)\, du\, ds \\ &= \varepsilon^{3/2} \int_0^{v/\varepsilon} \tilde{\Phi}^\varepsilon(s) \int_0^s \Phi^\varepsilon(u)\, du\, ds \qquad \text{for all} \quad 0 \leqslant v \leqslant 1. \end{aligned} \tag{A5.7}$$

Now define

$$S_1^\varepsilon(u) \triangleq \int_0^u \tilde{\Phi}^\varepsilon(s) \int_0^s \Phi^\varepsilon(w)\, dw\, ds \qquad \text{for all} \quad 0 \leqslant u \leqslant \varepsilon^{-1}. \tag{A5.8}$$

Then for all $0 \leqslant t \leqslant u \leqslant \varepsilon^{-1}$,

$$
\begin{aligned}
E & |S_1^\varepsilon(u) - S_1^\varepsilon(t)|^4 \\
& \leqslant 8E\left|\int_t^u \tilde{\Phi}^\varepsilon(s)\,ds \int_0^t \Phi^\varepsilon(w)\,dw\right|^4 + 8E\left|\int_t^u \tilde{\Phi}^\varepsilon(s) \int_t^s \Phi^\varepsilon(w)\,dw\,ds\right|^4 \\
& \leqslant 8d^4 \sum_{i=1,\,j=1}^{d} E\left|\int_t^u \tilde{\Phi}_{i,j}^\varepsilon(s)\,ds \int_0^t \Phi_j^\varepsilon(w)\,dw\right|^4 \\
& \quad + 8d^4 \sum_{i=1,\,j=1}^{d} E\left|\int_t^u \tilde{\Phi}_{i,j}^\varepsilon(s)\,ds \int_t^s \Phi_j^\varepsilon(w)\,dw\right|^4, \qquad \text{(A5.9)}
\end{aligned}
$$

where we have used the fact that $|\sum_{j=1}^d a_j|^4 \leqslant d^4 \sum_{j=1}^d |a_j|^4$ for real numbers $a_1 \cdots a_d$. We consider the first term on the right-hand side of (A5.9). By Cauchy–Schwarz and Fubini,

$$
\begin{aligned}
E & \left|\int_t^u \tilde{\Phi}_{i,j}^\varepsilon(s)\,ds \int_0^t \Phi_j^\varepsilon(w)\,dw\right|^4 \\
& \leqslant \left[\int_t^u \cdots \int_t^u |E\{\tilde{\Phi}_{i,j}^\varepsilon(s_1) \cdots \tilde{\Phi}_{i,j}^\varepsilon(s_8)\}|\,ds_8 \cdots ds_1 \right. \\
& \qquad \left. \times \int_0^t \cdots \int_0^t |E\{\Phi_j^\varepsilon(w_1) \cdots \Phi_j^\varepsilon(w_8)\}|\,dw_8 \cdots dw_1\right]^{1/2} \qquad \text{(A5.10)}
\end{aligned}
$$

for all $1 \leqslant i, j \leqslant d$. Now by (2.1), $|\tilde{\Phi}_{i,j}^\varepsilon(s)| \leqslant 2\bar{N}$ for $0 \leqslant s \leqslant \varepsilon^{-1}$, $1 \leqslant i, j \leqslant d$, and by Lemma A7.2(ii) of Appendix 7, $\|\Phi_j^\varepsilon(s)\|_{8+4\delta} \leqslant 2M + 2ND$ for $0 \leqslant s \leqslant \varepsilon^{-1}$, $j = 1, \ldots, d$. Therefore, by (A5.10) and Lemma A4(A) there exists some constant $c_5 > 0$ (depending only on M, N, D, and B_1', B_2', B_3', B_4' of Sections 2) such that

$$
E\left|\int_t^u \tilde{\Phi}_{i,j}^\varepsilon(s)\,ds \int_0^t \Phi_j^\varepsilon(v)\,dv\right|^4 \leqslant c_5(u-t)^2 t^2 \quad \text{for } 0 \leqslant t \leqslant u \leqslant \varepsilon^{-1} \qquad \text{(A5.11)}
$$

and $1 \leqslant i, j \leqslant d$. Now, for the second term of (A5.9), we have by Fubini,

$$
\begin{aligned}
E & \left(\int_t^u \tilde{\Phi}_{i,j}^\varepsilon(s) \int_t^s \Phi_j^\varepsilon(w)\,dw\,ds\right)^4 \\
& = E\left[\int_t^u \cdots \int_t^u \left\{\int_t^{s_1} \int_t^{s_2} \int_t^{s_3} \int_t^{s_4} \Phi_j^\varepsilon(w_1) \cdots \Phi_j^\varepsilon(w_4)\,dw_4 \cdots dw_1\right\}\right. \\
& \qquad \left. \times \tilde{\Phi}_{i,j}^\varepsilon(s_1) \cdots \tilde{\Phi}_{i,j}^\varepsilon(s_4)\,ds_4 \cdots ds_1\right] \\
& \leqslant \int_t^u \cdots \int_t^u |E\{\Phi_j^\varepsilon(w_1) \cdots \Phi_j^\varepsilon(w_4) \tilde{\Phi}_{i,j}^\varepsilon(s_1) \cdots \tilde{\Phi}_{i,j}^\varepsilon(s_4)\}| \\
& \qquad \times dw_4 \cdots dw_1\,ds_4 \cdots ds_1 \qquad \text{(A5.11)}
\end{aligned}
$$

for all $0 \leqslant t \leqslant u \leqslant \varepsilon^{-1}$ and $i, j = 1, ..., d$. Therefore, by Lemma A4(A) there exists some $c_6 > 0$ (depending only on M, N, D, d, and B'_1, B'_2, B'_3, B'_4) such that

$$E\left|\int_t^u \tilde{\Phi}^{\varepsilon}_{i,j}(s) \int_t^s \Phi^{\varepsilon}_j(w)\, dw\, ds\right|^4 \leqslant c_6(u-t)^4$$

$$\text{for all} \quad 0 \leqslant t \leqslant u \leqslant \varepsilon^{-1}, \quad i, j = 1, ..., d, \tag{A5.13}$$

and by (A5.9), (A5.11), and (A5.13) there exists some constant $c_7 > 0$ such that for all $0 \leqslant t \leqslant u \leqslant \varepsilon^{-1}$,

$$\begin{aligned} E\,|S^{\varepsilon}_1(u) - S^{\varepsilon}_1(t)|^4 &\leqslant c_7[(u-t)^2 t^2 + (u-t)^4] \\ &\leqslant c_7[(u^2-t^2)^2 + (u-t)^2\,(u+t)^2] \\ &= [h(t,u)]^2, \end{aligned} \tag{A5.14}$$

where $h(t, u) \triangleq \sqrt{8c_7} \cdot \int_t^u s\, ds$. By (A5.14), (A5.7), and Theorem A3.1 (with $v \triangleq 4$, $\gamma \triangleq 2$, $Q_t \triangleq S^{\varepsilon}_1(t)$), there exists an absolute constant $a > 0$ such that

$$\begin{aligned} E\,\|A^{\varepsilon}\|^4_C &= \varepsilon^6 E[\max_{0 \leqslant t \leqslant \varepsilon^{-1}} |S^{\varepsilon}_1(t) - S^{\varepsilon}_1(0)|^4] \\ &\leqslant \varepsilon^6 a[h(0, \varepsilon^{-1})]^2 = a\varepsilon^6 \cdot 2c_7 \varepsilon^{-4}. \end{aligned} \tag{A5.15}$$

The lemma follows by choosing $c_4 = 2ac_7$. ∎

LEMMA A5.3. *Let $B^{\varepsilon}(v)$ be defined as in* (3.22) *for* $0 \leqslant v \leqslant 1$. *Then under conditions* (C0)–(C5) *of Section* 2 *there exists a constant* $c_8 > 0$, *depending only on constants M, N, D, d, and $B'_1, ..., B'_4$ of Section* 2 *such that for all* $\varepsilon > 0$,

$$E\,\|B^{\varepsilon}\|^4_C \leqslant c_8 \varepsilon^2.$$

Proof. Fix $\varepsilon > 0$ and put $\Phi^{\varepsilon}(s) \triangleq \tilde{F}(x^0(\varepsilon s), s)$ and $\tilde{\Phi}^{\varepsilon}(s) \triangleq [(\partial F/\partial x)(x^0(\varepsilon s), s) - E(\partial F/\partial x)(x^0(\varepsilon s), s)]$ for $0 \leqslant s \leqslant \varepsilon^{-1}$. By a change of variables, (3.1), and then two more changes of variables,

$$\begin{aligned} B^{\varepsilon}(v) &= \int_0^v \left[\frac{\partial F}{\partial x}(x^0(s), s/\varepsilon) - E\frac{\partial F}{\partial x}(x^0(s), s/\varepsilon)\right] \int_0^s K^{\varepsilon}(s,u)\, W^{\varepsilon}_1(u)\, du\, ds \\ &= \varepsilon^{1/2} \int_0^{v/\varepsilon} \tilde{\Phi}^{\varepsilon}(s) \int_0^{\varepsilon s} K^{\varepsilon}(\varepsilon s, u) \int_0^u \tilde{F}(x^0(w), w/\varepsilon)\, dw\, du\, ds \\ &= \varepsilon^{3/2} \int_0^{v/\varepsilon} \tilde{\Phi}^{\varepsilon}(s) \int_0^s K^{\varepsilon}(\varepsilon s, \varepsilon u) \int_0^{u\varepsilon} \tilde{F}(x^0(w), w/\varepsilon)\, dw\, du\, ds \end{aligned}$$

$$= \varepsilon^{5/2} \int_0^{v/\varepsilon} \tilde{\Phi}^\varepsilon(s) \int_0^s K^\varepsilon(\varepsilon s, \varepsilon u) \int_0^u \Phi^\varepsilon(w)\, dw\, du\, ds$$

$$= \varepsilon^{5/2} \int_0^{v/\varepsilon} \tilde{\Phi}^\varepsilon(s) \int_0^s \psi^\varepsilon(s, w) \Phi^\varepsilon(w)\, dw\, ds \tag{A5.16}$$

for all $0 \leqslant v \leqslant 1$, where

$$\psi^\varepsilon(s, w) \triangleq \int_w^s K^\varepsilon(\varepsilon s, \varepsilon y)\, dy \tag{A5.17}$$

for $0 \leqslant w \leqslant s \leqslant \varepsilon^{-1}$. Define

$$S_2^\varepsilon(u) \triangleq \int_0^u \tilde{\Phi}^\varepsilon(s) \int_0^s \psi^\varepsilon(s, w) \Phi^\varepsilon(w)\, dw\, ds \tag{A5.18}$$

for $0 \leqslant u \leqslant \varepsilon^{-1}$ and $\omega \in \Omega$. Fixing $0 \leqslant t \leqslant u \leqslant \varepsilon^{-1}$, we have by (A5.18),

$$\begin{aligned} &E[|S_2^\varepsilon(u) - S_2^\varepsilon(t)|^4] \\ &\quad \leqslant 8E\left| \int_t^u \tilde{\Phi}^\varepsilon(s) \int_0^t \psi^\varepsilon(s, w) \Phi^\varepsilon(w)\, dw\, ds \right|^4 \\ &\qquad + 8E\left| \int_t^u \tilde{\Phi}^\varepsilon(s) \int_t^s \psi^\varepsilon(s, w) \Phi^\varepsilon(w)\, dw\, ds \right|^4 \\ &\quad \leqslant 8d^4 \sum_{i=1}^d \sum_{j=1}^d E\left| \int_t^u \tilde{\Phi}^\varepsilon_{i,j}(s) \int_0^t \sum_{k=1}^d \Phi_k^\varepsilon(w) \psi_{j,k}^\varepsilon(s, w)\, dw\, ds \right|^4 \\ &\qquad + 8d^4 \sum_{i=1}^d \sum_{j=1}^d E\left| \int_t^u \tilde{\Phi}^\varepsilon_{i,j}(s) \int_t^s \sum_{k=1}^d \Phi_k^\varepsilon(w) \psi_{j,k}^\varepsilon(s, w)\, dw\, ds \right|^4. \end{aligned} \tag{A5.19}$$

But, by Fubini, (A5.17), and (3.19) there exists constant $c_9 (= N^4 e^{8N})$ such that

$$\begin{aligned} &E\left(\int_t^u \tilde{\Phi}^\varepsilon_{i,j}(s) \int_t^s \sum_{k=1}^d \Phi_k^\varepsilon(w) \psi_{j,k}^\varepsilon(s, w)\, dw\, ds \right)^4 \\ &\quad = E\left\{ \int_t^u \cdots \int_t^u \left\{ \int_t^{s_1} \int_t^{s_2} \int_t^{s_3} \int_t^{s_4} \right.\right. \\ &\qquad \sum_{k_1=1}^d \cdots \sum_{k_4=1}^d \tilde{\Phi}^\varepsilon_{i,j}(s_1) \cdots \tilde{\Phi}^\varepsilon_{i,j}(s_4) \Phi^\varepsilon_{k_1}(w_1) \cdots \Phi^\varepsilon_{k_4}(w_4) \\ &\qquad \left.\left. \times \psi^\varepsilon_{j,k_1}(s_1, w_1) \cdots \psi^\varepsilon_{j,k_4}(s_4, w_4)\, dw_4\, dw_3\, dw_2\, dw_1 \right\} ds_4\, ds_3\, ds_2\, ds_1 \right\} \end{aligned}$$

$$\leqslant \int_t^u \cdots \int_t^u \int_t^{s_1} \cdots \int_t^{s_4} \sum_{k_1=1}^d \cdots \sum_{k_4=1}^d$$
$$|E\{\tilde{\Phi}_{i,j}^\varepsilon(s_1) \cdots \tilde{\Phi}_{i,j}^\varepsilon(s_4) \Phi_{k_1}^\varepsilon(w_1) \cdots \Phi_{k_4}^\varepsilon(w_4)\}$$
$$\times \psi_{j,k_1}^\varepsilon(s_1, w_1) \cdots \psi_{j,k_4}^\varepsilon(s_4, w_4)|\, dw_4 \cdots dw_1\, ds_4 \cdots ds_1$$
$$\leqslant c_9 (u-t)^4 \sum_{k_1=1}^d \cdots \sum_{k_4=1}^d \int_t^u \cdots \int_t^u$$
$$|E\{\tilde{\Phi}_{i,j}^\varepsilon(s_1) \cdots \tilde{\Phi}_{i,j}^\varepsilon(s_4) \Phi_{k_1}^\varepsilon(w_1) \cdots \Phi_{k_4}^\varepsilon(w_4)\}|$$
$$\times dw_4 \cdots dw_1\, ds_4 \cdots ds_1. \qquad \text{(A5.20)}$$

Therefore, by (A5.20) and Lemma A4(A) there exists constant $c_{10} > 0$ (depending only on constants M, N, D, d, and B_1', B_2', B_3', B_4' of Section 2) such that

$$E\left(\int_t^u \tilde{\Phi}_{i,j}^\varepsilon(s) \int_t^s \sum_{k=1}^d \Phi_k^\varepsilon(w) \psi_{j,k}^\varepsilon(s, w)\, dw\, ds\right)^4 \leqslant c_{10}(u-t)^8. \qquad \text{(A5.21)}$$

Similarly to (A5.20), there is some $c_{11} (= N^4 e^{8N})$ such that

$$E\left(\int_t^u \tilde{\Phi}_{i,j}^\varepsilon(s) \int_0^t \sum_{k=1}^d \Phi_k^\varepsilon(w) \psi_{j,k}^\varepsilon(s, w)\, dw\, ds\right)^4$$
$$\leqslant c_{11} u^4 \sum_{k_1=1}^d \cdots \sum_{k_4=1}^d \int_t^u \cdots \int_t^u \int_0^t \cdots \int_0^t$$
$$|E\{\tilde{\Phi}_{i,j}^\varepsilon(s_1) \cdots \tilde{\Phi}_{i,j}^\varepsilon(s_4) \Phi_{k_1}^\varepsilon(w_1) \cdots \Phi_{k_4}^\varepsilon(w_4)\}|$$
$$\times dw_4 \cdots dw_1\, ds_4 \cdots ds_1. \qquad \text{(A5.22)}$$

Therefore, by (A5.22) and Lemma A4(C), there exists constant $c_{12} > 0$, depending only on M, N, D, d, and B_1', B_2', B_3', B_4' such that

$$E\left(\int_t^u \tilde{\Phi}_{i,j}^\varepsilon(s) \int_0^t \sum_{k=1}^d \Phi_k^\varepsilon(w)\, \psi_{j,k}^\varepsilon(s, w)\, dw\, ds\right)^4 \leqslant c_{12} u^4 (u-t)^2 t^2. \qquad \text{(A5.23)}$$

By (A5.19), (A5.21), and (A5.23) there exists a constant $c_{13} > 0$ (depending only on M, N, D, d, and B_1', B_2', B_2', B_3', B_4') such that

$$\begin{aligned} E[|S_2^\varepsilon(u) - S_2^\varepsilon(t)|^4] &\leqslant c_{13}[(u-t)^8 + u^4 t^2 (u-t)^2] \\ &\leqslant c_{13}[((u-t)(u+t))^4 + (u^3 t - u^2 t^2)^2] \\ &\leqslant c_{13}[((u^2 - t^2)(u^2 + t^2))^2 + (u^4 - t^4)^2] \\ &= [h(t, u)]^2 \end{aligned} \qquad \text{(A5.24)}$$

for all $0 \leqslant t \leqslant u \leqslant \varepsilon^{-1}$, where

$$h(t, u) \triangleq \sqrt{32c_{13}} \int_t^u s^3\, ds \qquad \text{for all} \quad 0 \leqslant t \leqslant u \leqslant \varepsilon^{-1}. \tag{A5.25}$$

By (A5.24), (A5.25), and Theorem A3.1 (with $v \triangleq 4$, $\gamma \triangleq 2$, $Q_t \triangleq S_2^\varepsilon(t)$) there exists $c_{14} > 0$ depending only on M, N, D, d, and B_1', B_2', B_3', B_4' such that

$$E[\max_{0 \leqslant t \leqslant \varepsilon^{-1}} |S_2^\varepsilon(t)|^4] \leqslant c_{14} \varepsilon^{-8}. \tag{A5.26}$$

Therefore by (A5.16), (A5.18), and (A5.26),

$$E\, \|B^\varepsilon\|_C^4 = (\varepsilon^{5/2})^4\, E[\max_{0 \leqslant t \leqslant \varepsilon^{-1}} |S_2^\varepsilon(t)|^4] \leqslant c_{14} \varepsilon^2$$

$$\text{for all} \quad \varepsilon > 0. \quad \blacksquare \tag{A5.27}$$

APPENDIX 6: A Functional CLT for W_1^ε

In this appendix a functional central limit theorem with error term is developed for the process $W_1^\varepsilon(\cdot)$ defined in (3.1). The main result of this appendix, Lemma A6.1, is used in Eq. (3.28). Lemmas A6.2 and A6.3 are subsidiary technical results; Lemma A6.2 is used to establish Lemma A6.1, while Lemma A6.3 supports the proof of Lemma A6.2.

Lemma A6.1. *Under the conditions* (C0)–(C6) *of Section* 2, *there are constants* $c_1 > 0$, $\rho > 0$, *and* $\varepsilon_0 \in (0, 1]$ *such that*

$$\Pi(\mathscr{L}(W_1^\varepsilon), \mathscr{L}(\hat{W}^0)) \leqslant c_1 \varepsilon^\rho \qquad \textit{for all} \quad 0 < \varepsilon \leqslant \varepsilon_0,$$

where $W_1^\varepsilon(\cdot)$ *and* $\hat{W}^0(\cdot)$ *are defined in Eqs.* (3.1) *and* (2.14), *respectively.*

Remark A6.1. In the above formulation, ρ can be taken to be $3\mu/152$, where μ is a constant defined in Lemma A6.2 which follows.

Proof. For each $k = 1, 2, 3, \ldots$ and $\varepsilon > 0$ define the process $\{W_{1,k}^\varepsilon(\tau),\ 0 \leqslant \tau \leqslant 1\}$ on $(\Omega, \mathscr{F}, P)$ by

$$W_{1,k}^\varepsilon(\tau) \triangleq W_1^\varepsilon(\tau) \qquad \text{for} \quad \tau = i/k, \quad i = 0, \ldots, k, \tag{A6.1}$$

and $W_{1,k}^\varepsilon(\tau)$ is given by linear interpolation over the intervals $[i/k, (i+1)/k]$ for $i = 0, 1, \ldots, k-1$. Define the process $\{\hat{W}_k^0(\tau),\ 0 \leqslant \tau \leqslant 1\}$ on $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$ in terms of $\{\hat{W}^0(\tau),\ 0 \leqslant \tau \leqslant 1\}$ in a similar manner.

By Lemma A2.3 and Lemma A6.2 there exist constants $c_2 > 0$, $\mu > 0$, and positive integer K_0 such that

$$\Pi(\mathscr{L}(W^{\varepsilon}_{1,k}), \mathscr{L}(\hat{W}^0_k)) \leqslant c_2 k^{19/6} \varepsilon^{\mu} \qquad \text{for all} \quad k \geqslant K_0, \quad 0 < \varepsilon \leqslant k^{-19/3\mu}. \tag{A6.2}$$

Now find the $\varepsilon_0 \in (0, 1]$ such that

$$\varepsilon_0^{-3\mu/19} = K_0, \tag{A6.3}$$

fix $0 < \varepsilon \leqslant \varepsilon_0$ (to remain fixed throughout the remainder of this proof), and define

$$k(\varepsilon) \triangleq [\varepsilon^{-3\mu/19}]. \tag{A6.4}$$

Now, by the triangle inequality,

$$\begin{aligned} \Pi(\mathscr{L}(W^{\varepsilon}_1), \mathscr{L}(\hat{W}^0)) \leqslant \Pi(\mathscr{L}(W^{\varepsilon}_1), \mathscr{L}(W^{\varepsilon}_{1,k(\varepsilon)})) + \Pi(\mathscr{L}(W^{\varepsilon}_{1,k(\varepsilon)}), \mathscr{L}(\hat{W}^0_{k(\varepsilon)})) \\ + \Pi(\mathscr{L}(\hat{W}^0_{k(\varepsilon)}), \mathscr{L}(\hat{W}^0)). \end{aligned} \tag{A6.5}$$

Consider the second term on the right-hand side of (A6.5) first. By (A6.4) and (A6.3), $\varepsilon^{-3\mu/19} \geqslant k(\varepsilon) \geqslant K_0$ and so by (A6.2),

$$\Pi(\mathscr{L}(W^{\varepsilon}_{1,k(\varepsilon)}), \mathscr{L}(\hat{W}^0_{k(\varepsilon)})) \leqslant c_2 [\varepsilon^{-3\mu/19}]^{19/6} \varepsilon^{\mu} \leqslant c_2 \varepsilon^{\mu/2}. \tag{A6.6}$$

Now consider the first term on the right of (A6.5). By the definition of $\{W^{\varepsilon}_{1,k(\varepsilon)}(\tau)\}$ and (3.1),

$$\begin{aligned} &\max_{i/k(\varepsilon) \leqslant \tau \leqslant (i+1)/k(\varepsilon)} |W^{\varepsilon}_1(\tau) - W^{\varepsilon}_{1,k(\varepsilon)}(\tau)| \\ &\leqslant \max_{i/k(\varepsilon) \leqslant \tau \leqslant (i+1)/k(\varepsilon)} \left\{ \left| W^{\varepsilon}_1(\tau) - W^{\varepsilon}_1\left(\frac{i}{k(\varepsilon)}\right) \right| \right. \\ &\qquad \left. + (k(\varepsilon)\tau - i) \left| W^{\varepsilon}_1\left(\frac{i+1}{k(\varepsilon)}\right) - W^{\varepsilon}_1\left(\frac{i}{k(\varepsilon)}\right) \right| \right\} \\ &\leqslant 2 \cdot \max_{i/k(\varepsilon) \leqslant \tau \leqslant (i+1)/k(\varepsilon)} \left| W^{\varepsilon}_1(\tau) - W^{\varepsilon}_1\left(\frac{i}{k(\varepsilon)}\right) \right| \\ &\leqslant 2\varepsilon^{1/2} \max_{i/k(\varepsilon) \leqslant \tau \leqslant (i+1)/k(\varepsilon)} \left| \int_{i/\varepsilon k(\varepsilon)}^{\tau/\varepsilon} \tilde{F}(x^0(\varepsilon s), s)\, ds \right| \end{aligned} \tag{A6.7}$$

for all $\omega \in \Omega$ and $i = 0, ..., k(\varepsilon) - 1$. By Lemma A7.2(ii),

$$\sup_{0 \leqslant s \leqslant \varepsilon^{-1}} \|\tilde{F}_m(x^0(\varepsilon s), s)\|_{4+2\delta} \leqslant 2M + 2ND \qquad \text{for all} \quad m = 1, ..., d \tag{A6.8}$$

and, so by Lemma A4(A) there exists $c_3 > 0$ (depending only on constants, M, N, D, d, and B'_1, B'_2 of Section 2) such that for any $0 \leqslant t < u \leqslant \varepsilon^{-1}$,

$$E\,|\tilde{S}_u - \tilde{S}_t|^4 = E\left|\int_t^u \tilde{F}(x^0(\varepsilon s), s)\,ds\right|^4 \leqslant c_3(u-t)^2 = [h(t,u)]^2, \qquad \text{(A6.9)}$$

where $h(t,u) \triangleq \sqrt{c_3}\int_t^u ds$ and

$$\tilde{S}_u \triangleq \int_0^u \tilde{F}(x^0(\varepsilon s), s)\,ds. \qquad \text{(A6.10)}$$

By (A6.7), a substitution $t = \tau/\varepsilon$, (A6.10), (A6.9), and Theorem A3.1 (with $\gamma \triangleq 4$, $v \triangleq 2$, and $Q_t \triangleq \tilde{S}_t$) there exists absolute constant $c_4 > 0$,

$$\begin{aligned} &E[\max_{i/k(\varepsilon) \leqslant \tau \leqslant (i+1)/k(\varepsilon)} |W_1^\varepsilon(\tau) - W_{1,k(\varepsilon)}^\varepsilon(\tau)|^4] \\ &\quad \leqslant 16\varepsilon^2 E[\max_{i/\varepsilon k(\varepsilon) \leqslant t \leqslant (i+1)/\varepsilon k(\varepsilon)} |\tilde{S}_t - \tilde{S}_{i/\varepsilon k(\varepsilon)}|^4] \\ &\quad \leqslant 16\varepsilon^2 c_4\left[h\left(\frac{i}{\varepsilon k(\varepsilon)}, \frac{i+1}{\varepsilon k(\varepsilon)}\right)\right]^2 = 16 c_4 c_3 k(\varepsilon)^{-2} \end{aligned} \qquad \text{(A6.11)}$$

for $i = 0, 1, ..., k(\varepsilon) - 1$. Thus, letting $c' = 32c_4c_3$, we have by Chebyshev and (A6.11),

$$\begin{aligned} &P(\|W_1^\varepsilon - W_{1,k(\varepsilon)}^\varepsilon\|_C \geqslant \varepsilon^{3\mu/152} + c'\varepsilon^{3\mu/38}) \\ &\quad \leqslant \sum_{i=0}^{k(\varepsilon)-1} P(\max_{i/k(\varepsilon) \leqslant \tau \leqslant (i+1)/k(\varepsilon)} |W_1^\varepsilon(\tau) - W_{1,k(\varepsilon)}^\varepsilon(\tau)| \geqslant \varepsilon^{3\mu/152}) \\ &\quad \leqslant k(\varepsilon)\frac{16c_4c_3}{k^2(\varepsilon)\varepsilon^{3\mu/38}} \leqslant c'\varepsilon^{3\mu/38} + \varepsilon^{3\mu/152}. \end{aligned} \qquad \text{(A6.12)}$$

Finally, by Lemma A2.2(a) and (A6.12) there exists $c_5 > 0$ such that

$$\Pi(\mathscr{L}(W_1^\varepsilon), \mathscr{L}(W_{1,k(\varepsilon)}^\varepsilon)) < c_5\varepsilon^{3\mu/152}. \qquad \text{(A6.13)}$$

Now, we consider the third term on the right of (A6.5). As in (A6.7),

$$\begin{aligned} &\max_{i/k(\varepsilon) \leqslant \tau \leqslant (i+1)/k(\varepsilon)} |\hat{W}^0(\tau) - \hat{W}_{k(\varepsilon)}^0(\tau)| \\ &\quad \leqslant 2 \max_{i/k(\varepsilon) \leqslant \tau \leqslant (i+1)/k(\varepsilon)} \left|\hat{W}^0(\tau) - \hat{W}^0\left(\frac{i}{k(\varepsilon)}\right)\right| \end{aligned} \qquad \text{(A6.14)}$$

for all $\hat{\omega} \in \hat{\Omega}$ and $i = 0, ..., k(\varepsilon) - 1$. Moreover, by (2.14), $\hat{W}^0(\cdot)$ has continuous sample paths on $[0, 1]$ and for $0 \leqslant \tau < \tau' \leqslant 1$,

$$\hat{W}^0(\tau') - \hat{W}^0(\tau) \sim \mathcal{N}\left(0, \int_\tau^{\tau'} A(x^0(s))\, ds\right). \tag{A6.15}$$

By Lemma A7.1, $A_{m,m}(x)$ is bounded for $|x| \leqslant D$ so there exists constant $c_6 > 0$ such that

$$\hat{E}\, |\hat{W}^0(\tau) - \hat{W}^0(\tau')|^4 \leqslant \sum_{m=1}^{d} \hat{E}\, |\hat{W}^0_m(\tau) - \hat{W}^0_m(\tau')|^4 \leqslant c_6\, |\tau - \tau'|^2 \tag{A6.16}$$

for all $0 \leqslant \tau,\ \tau' \leqslant 1$. If we define

$$h(\tau, \tau') \triangleq \sqrt{c_6} \int_\tau^{\tau'} ds \qquad \text{for all} \quad 0 \leqslant \tau \leqslant \tau' \leqslant 1, \tag{A6.17}$$

then by (A6.14), (A6.16), and Theorem A3.1 (with $v \triangleq 4$, $\gamma \triangleq 2$, $Q_\tau \triangleq \hat{W}^0(\tau)$) there exists constant $c_7 > 0$ such that

$$\begin{aligned} &\hat{E}[\max_{i/k(\varepsilon) \leqslant \tau \leqslant (i+1)/k(\varepsilon)} |\hat{W}^0(\tau) - \hat{W}^0_{k(\varepsilon)}(\tau)|^4] \\ &\quad \leqslant 16 \hat{E}\left[\max_{i/k(\varepsilon) \leqslant \tau \leqslant (i+1)/k(\varepsilon)} \left|\hat{W}^0(\tau) - \hat{W}^0\left(\frac{i}{k(\varepsilon)}\right)\right|^4\right] \\ &\quad \leqslant \frac{c_7}{k^2(\varepsilon)}. \end{aligned} \tag{A6.18}$$

Thus, as in (A6.12),

$$\hat{P}[\| \hat{W}^0(\tau) - \hat{W}^0_{k(\varepsilon)}(\tau) \|_C \geqslant \varepsilon^{3\mu/152} + 2c_7 \varepsilon^{3\mu/38}] \leqslant 2c_7 \varepsilon^{3\mu/38} + \varepsilon^{3\mu/152}. \tag{A6.19}$$

Thus by Lemma A2.2 there exists $c_8 > 0$ such that

$$\Pi(\mathscr{L}(\hat{W}^0), \mathscr{L}(\hat{W}^0_{k(\varepsilon)})) < c_8 \varepsilon^{3\mu/152} \qquad \text{for} \quad 0 < \varepsilon \leqslant \varepsilon_0. \tag{A6.20}$$

The lemma follows from (A6.5), (A6.6), (A6.13), and (A6.20). ∎

The next lemma A6.2 is a central limit theorem with an error term for vectors of evenly displaced samples of the processes $W^\varepsilon_1(\cdot)$ and $W \pm {}^0(\cdot)$ defined in (3.1) and (2.14), respectively. It is used in conjunction with

Lemma A2.3 at line (A6.2) of Lemma A6.1. For ease of formulation and development of Lemma A6.2, we define the kd-variate random vectors,

$$\Xi_k^\varepsilon \triangleq \begin{bmatrix} W_1^\varepsilon(1/k) \\ W_1^\varepsilon(2/k) \\ \vdots \\ W_1^\varepsilon(1) \end{bmatrix} \quad \text{and} \quad \hat{\Xi}_k^0 \triangleq \begin{bmatrix} \hat{W}^0(1/k) \\ \hat{W}^0(2/k) \\ \vdots \\ \hat{W}^0(1) \end{bmatrix}, \tag{A6.21}$$

where $k = 1, 2, 3, \ldots$ and $\varepsilon > 0$.

LEMMA A6.2. *Under conditions* (C0)–(C6) *of Section* 2 *and with the notation of* (A6.21), *there exist constants* $c' > 0$ *and* $\mu \triangleq \min\{1/33, \chi/16\}$ *and a positive integer* K_0 *such that*

$$\Pi_\infty^{kd}(\mathscr{L}(\Xi_k^\varepsilon), \mathscr{L}(\hat{\Xi}_k^0)) \leqslant c' k^{19/6} \varepsilon^\mu \qquad \textit{for all} \quad 0 < \varepsilon \leqslant k^{-19/3\mu}, \quad k \geqslant K_0.$$

Here $0 < \chi \leqslant 1$ *is the constant of condition* (C6) *in Section* 2.

Proof. The following definitions are used (in all three parts of the proof) to help capture the mixing properties of process $W_1^\varepsilon(\cdot)$. For any $\varepsilon > 0$, $k = 1, 2, ..$ let

$$p(\varepsilon, k) \triangleq k^{-1}\varepsilon^{-3/4}, \qquad q(\varepsilon, k) \triangleq k^{-1}\varepsilon^{-1/4}, \tag{A6.22}$$

and

$$l(\varepsilon, k) \triangleq \left[\frac{k^{-1}\varepsilon^{-1}}{p(\varepsilon, k) + q(\varepsilon, k)}\right]. \tag{A6.23}$$

Clearly by (A6.22) and (A6.23),

$$\tfrac{1}{4}\varepsilon^{-1/4} \leqslant l(\varepsilon, k) \leqslant \varepsilon^{-1/4} \qquad \text{for all} \quad 0 < \varepsilon \leqslant \tfrac{1}{16}, \quad k = 1, 2, \ldots. \tag{A6.24}$$

Henceforth, for ease of notation, we will drop the explicit indication that p, q, and l depend on ε and k. Now in preparation for more definitions, fix an $\varepsilon > 0$ and a positive integer k and for all integers i, j such that $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant k$; let $H_{i,j}^{\varepsilon,k}$ (long block) and $I_{i,j}^{\varepsilon,k}$ (short block) be the intervals of length p and q respectively given by

$$\begin{aligned} H_{i,j}^{\varepsilon,k} \triangleq [&(j-1)\,\varepsilon^{-1}k^{-1} + (i-1)(p+q), \\ &(j-1)\,\varepsilon^{-1}k^{-1} + (i-1)(p+q) + p] \end{aligned} \tag{A6.25}$$

and

$$\begin{aligned} I_{i,j}^{\varepsilon,k} \triangleq [&(j-1)\,\varepsilon^{-1}k^{-1} + (i-1)(p+q) + p, \\ &(j-1)\,\varepsilon^{-1}k^{-1}q + i(p+q)]. \end{aligned} \tag{A6.26}$$

Furthermore, for each integer j such that $1 \leqslant j \leqslant k$, let $I^{\varepsilon,k}_{l+1,j}$ (leftover error block) be the interval of length $k^{-1}\varepsilon^{-1} - l(p+q) \leqslant p+q$:

$$I^{\varepsilon,k}_{l+1,j} \triangleq [(j-1)\varepsilon^{-1}k^{-1} + l(p+q), j\varepsilon^{-1}k^{-1}]. \tag{A.27}$$

Clearly, the adjacent intervals $H^{\varepsilon,k}_{1,j}, I^{\varepsilon,k}_{1,j}, H^{\varepsilon,k}_{2,j}, I^{\varepsilon,k}_{2,j}, ..., H^{\varepsilon,k}_{l,j}, I^{\varepsilon,k}_{l,j}, I^{\varepsilon,k}_{l+1,j}$ fill up the interval $[(j-1)\varepsilon^{-1}k^{-1}, j\varepsilon^{-1}k^{-1}]$ for $j = 1, ..., k$. Now for $j = 1, ..., k$, we define the random d-vectors,

$$Y^{\varepsilon,k}_{i,j} \triangleq \int_{H^{\varepsilon,k}_{i,j}} \tilde{F}(x^0(\varepsilon s), s)\, ds, \qquad \text{for} \quad i = 1, ..., l, \tag{A6.28}$$

and

$$Z^{\varepsilon,k}_{i,j} \triangleq \int_{I^{\varepsilon,k}_{i,j}} \tilde{F}(x^0(\varepsilon s), s)\, ds \qquad \text{for} \quad i = 1, ..., l+1. \tag{A6.29}$$

Finally, for $i = 1, ..., l$, we define on $(\Omega, \mathscr{F}, P)$ the random kd-vectors:

$$\begin{aligned}
\tilde{Y}^{\varepsilon,k}_{i,1} &\triangleq ((Y^{\varepsilon,k}_{i,1})^T (Y^{\varepsilon,k}_{i,1})^T (Y^{\varepsilon,k}_{i,1})^T \cdots (Y^{\varepsilon,k}_{i,1})^T)^T \\
\tilde{Y}^{\varepsilon,k}_{i,2} &\triangleq (0 \cdots 0\, (Y^{\varepsilon,k}_{i,2})^T (Y^{\varepsilon,k}_{i,2})^T \cdots (Y^{\varepsilon,k}_{i,2})^T)^T \\
\tilde{Y}^{\varepsilon,k}_{i,3} &\triangleq (0 \cdots 0\, 0 \cdots 0\, (Y^{\varepsilon,k}_{i,3})^T \cdots (Y^{\varepsilon,k}_{i,3})^T)^T \\
&\vdots \\
\tilde{Y}^{\varepsilon,k}_{i,k} &\triangleq (0 \cdots 0\, 0 \cdots 0 \cdots 0 \cdots 0\, (Y^{\varepsilon,k}_{i,k})^T)^T.
\end{aligned} \tag{A6.30}$$

By the triangle inequality, for any $0 < \varepsilon \leqslant 1$, $k = 1, 2, ...,$

$$\begin{aligned}
&\Pi^{kd}_\infty(\mathscr{L}(\Xi^\varepsilon_k), \mathscr{L}(\hat{\Xi}^0_k)) \\
&\quad \leqslant \Pi^{kd}_\infty\left(\mathscr{L}(\Xi^\varepsilon_k), \mathscr{L}\left(\varepsilon^{1/2} \sum_{j=1}^{k} \sum_{i=1}^{l} \tilde{Y}^{\varepsilon,k}_{i,j}\right)\right. \\
&\qquad + \Pi^{kd}_\infty\left(\mathscr{L}\left(\varepsilon^{1/2} \sum_{j=1}^{k} \sum_{i=1}^{l} \tilde{Y}^{\varepsilon,k}_{i,j}\right), \mathscr{N}\left(0, \varepsilon \sum_{j=1}^{k} \sum_{i=1}^{l} \operatorname{cov}(\tilde{Y}^{\varepsilon,k}_{i,j})\right)\right) \\
&\qquad + \Pi^{kd}_\infty\left(\mathscr{N}\left(0, \varepsilon \sum_{j=1}^{k} \sum_{i=1}^{l} \operatorname{cov}(\tilde{Y}^{\varepsilon k}_{i,j})\right), \mathscr{L}(\hat{\Xi}^0_k)\right),
\end{aligned} \tag{A6.31}$$

where $\mathscr{N}(0, Q)$ is the zero mean, covariance Q normal distribution on $\mathfrak{R}^{kd}$. We consider the first term on the right-hand side of (A6.31). Fix an

$\varepsilon \in (0, \frac{1}{16}]$, a positive integer k, and define for each $i = 1, 2, ..., l+1$ the random kd-vectors,

$$\tilde{Z}_i^{\varepsilon,k} \triangleq \begin{bmatrix} Z_{i,1}^{\varepsilon,k} \\ Z_{i,1}^{\varepsilon,k} + Z_{i,2}^{\varepsilon,k} \\ Z_{i,1}^{\varepsilon,k} + Z_{i,2}^{\varepsilon,k} + Z_{i,3}^{\varepsilon,k} \\ \vdots \\ Z_{i,1}^{\varepsilon,k} + Z_{i,2}^{\varepsilon,k} + Z_{i,3}^{\varepsilon,k} + \cdots + Z_{i,k}^{\varepsilon,k} \end{bmatrix}. \tag{A6.32}$$

By (A6.28), (A6.29), (A6.25), (A6.26), (A6.27), and (3.1) for each integer $j = 1, 2, ..., k$,

$$\varepsilon^{1/2} \left\{ \sum_{i=1}^{l} Y_{i,j}^{\varepsilon,k} + \sum_{i=1}^{l+1} Z_{i,j}^{\varepsilon,k} \right\} = W_1^{\varepsilon}(jk^{-1}) - W_1^{\varepsilon}((j-1)k^{-1}) \tag{A6.33}$$

and, therefore, by (A6.30), (A6.32), (A6.33), and (A6.21),

$$\varepsilon^{1/2} \sum_{j=1}^{k} \sum_{i=1}^{l} \tilde{Y}_{i,j}^{\varepsilon,k} + \varepsilon^{1/2} \sum_{i=1}^{l+1} \tilde{Z}_i^{\varepsilon,k} = \Xi_k^{\varepsilon}. \tag{A6.34}$$

By (A6.34) and Minkowski's inequality,

$$\left\| \Xi_k^{\varepsilon} - \varepsilon^{1/2} \sum_{j=1}^{k} \sum_{i=1}^{l} \tilde{Y}_{i,j}^{\varepsilon,k} \right\|_2 \leqslant \varepsilon^{1/2} \sum_{i=1}^{l+1} \|\tilde{Z}_i^{\varepsilon,k}\|_2. \tag{A6.35}$$

Moreover, for $i = 1, 2, ..., l+1$,

$$E\,|\tilde{Z}_i^{\varepsilon,k}|^2 = E\left[\max_{1 \leqslant m \leqslant k} \left| \sum_{j=1}^{m} Z_{i,j}^{\varepsilon,k} \right|^2 \right] \leqslant \sum_{m=1}^{k} E \left| \sum_{j=1}^{m} Z_{i,j}^{\varepsilon,k} \right|^2. \tag{A6.36}$$

Now for nonnegative constants $\{a_i\}_{i=1}^n$ and $0 < p < 1$, we have $(\sum_{i=1}^n a_i)^p \leqslant \sum_{i=1}^n a_i^p$ (see, e.g., Lemma 3.1 of Longnecker and Serfling [20]). Thus, by (A6.36) and Minkowski,

$$\|\tilde{Z}_i^{\varepsilon,k}\|_2 \leqslant \sum_{m=1}^{k} \left\| \sum_{j=1}^{m} Z_{i,j}^{\varepsilon,k} \right\|_2 \leqslant k \sum_{m=1}^{k} \|Z_{i,m}^{\varepsilon,k}\|_2. \tag{A6.37}$$

Now, by Lemma A4(A) there exists $c_8 > 0$, depending only on constants N, M, D, d, and B_1' of Section 2 such that for $j = 1, 2, ..., k$,

$$\|Z_{i,j}^{\varepsilon,k}\|_2 = \left\{ E\left[\left| \int_{I_{i,j}^{\varepsilon,k}} \tilde{F}(x^0(\varepsilon s), s)\, ds \right|^2 \right] \right\}^{1/2}$$

$$\leqslant \begin{cases} c_8 q^{1/2} & \text{for} \quad i = 1, ..., l, \\ c_8 (p+q)^{1/2} & \text{for} \quad i = l+1. \end{cases} \tag{A6.38}$$

Thus by (A6.35), (A6.37), (A6.38), (A6.22), and (A6.24),

$$\left\| \Xi_k^\varepsilon - \varepsilon^{1/2} \sum_{j=1}^{k} \sum_{i=1}^{l} \tilde{Y}_{i,j}^{\varepsilon,k} \right\|_2 \leqslant \varepsilon^{1/2} l k^2 c_8 q^{1/2} + \varepsilon^{1/2} k^2 c_8 (p+q)^{1/2}$$
$$< c_8 \{ \varepsilon^{1/8} k^{3/2} + \sqrt{2}\, \varepsilon^{1/8} k^{3/2} \} \qquad \text{(A6.39)}$$

and, therefore, by Lemma A2.2(b) (with $S \triangleq \Re^{kd}$, $\rho \triangleq |\cdot|$, and $r \triangleq 2$) there exists $c_9 > 0$ such that

$$\Pi_\infty^{kd} \left(\mathscr{L}(\Xi_k^\varepsilon), \mathscr{L}\left(\varepsilon^{1/2} \sum_{j=1}^{k} \sum_{i=1}^{l} \tilde{Y}_{i,j}^{\varepsilon,k} \right) \right) < c_9 \varepsilon^{1/12} k \qquad \text{(A6.40)}$$

for all $0 < \varepsilon \leqslant \frac{1}{16}$ and $k = 1, 2, 3, \ldots$.

Now consider the second term of (A6.31). Fix an integer $k \geqslant 2$, any $\varepsilon \in (0, k^{-4}]$, and define

$$\tilde{X}_{i,j}^{\varepsilon,k} \triangleq (kl\varepsilon)^{1/2} \tilde{Y}_{i,j}^{\varepsilon,k} \qquad \text{for all} \quad i = 1, \ldots, l, \quad j = 1, \ldots, k. \qquad \text{(A6.41)}$$

Then, since $E\tilde{Y}_{i,j}^{\varepsilon,k} = 0$,

$$\varepsilon \sum_{j=1}^{k} \sum_{i=1}^{l} \operatorname{cov}(\tilde{Y}_{i,j}^{\varepsilon,k}) = \varepsilon \sum_{j=1}^{k} \sum_{i=1}^{l} E[(\tilde{Y}_{i,j}^{\varepsilon,k})(\tilde{Y}_{i,j}^{\varepsilon,k})^T]$$
$$= (kl)^{-1} \sum_{j=1}^{k} \sum_{i=1}^{l} \operatorname{cov}(\tilde{X}_{i,j}^{\varepsilon,k}) \qquad \text{(A6.42)}$$

and by (A6.30) for any integers i, j such that $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant k$,

$$E\,|\hat{X}_{i,j}^{\varepsilon,k}|_2^3 = (kl\varepsilon)^{3/2} \cdot (k-j+1)^{3/2} \cdot E\,|Y_{i,j}^{\varepsilon,k}|_2^3 \leqslant (k^2 l\varepsilon)^{3/2} \cdot E\,|Y_{i,j}^{\varepsilon,k}|_2^3, \qquad \text{(A6.43)}$$

where $|\cdot|_2$ is the Euclidean norm in $\Re^d$. Now $|x|_2^4 \leqslant d^2 \sum_{i=1}^d x_i^4$ for $x \in \Re^d$. Thus by Hölder's inequality, (A6.28), Lemma A4(A), (A6.25), and (A6.22) there exists constant $c_{10} > 0$ (depending only on constants M, N, D, d, and B_1', B_2' of Section 2) such that for any integers $i = 1, \ldots, l$, $j = 1, \ldots, k$,

$$E\,|Y_{i,j}^{\varepsilon,k}|_2^3 \leqslant \left\{ E \left| \int_{H_{i,j}^{\varepsilon,k}} \tilde{F}(x^0(\varepsilon s), s)\, ds \right|_2^4 \right\}^{3/4} \leqslant c_{10} \{ k^{-2} \varepsilon^{-3/2} \}^{3/4}$$
$$= c_{10} k^{-3/2} \varepsilon^{-9/8}. \qquad \text{(A6.44)}$$

By (A6.43), (A6.44), and (A6.24),

$$E\,|\tilde{X}_{i,j}^{\varepsilon,k}|_2^3 \leqslant c_{10} k^{3/2} \qquad \text{for all} \quad i = 1, \ldots, l, \quad j = 1, \ldots, k \qquad \text{(A6.45)}$$

and by Fubini and (A6.25), the components of $\tilde{X}^{\varepsilon,k}_{i,j}$,

$$(\varepsilon lk)^{1/2}\int_{H^{\varepsilon,k}_{i,j}} \tilde{F}(x^0(\varepsilon s), s)\, ds \text{ are } \mathscr{F}^{(j-1)\varepsilon^{-1}k^{-1}+(i-1)(p+q)+p}_{(j-1)\varepsilon^{-1}k^{-1}+(i-1)(p+q)}\text{-measurable} \tag{A6.46}$$

for all $i=1,\ldots,l$, and $j=1,2,\ldots,k$. Thus by Lemma A2.4, (A6.41), (A6.42), and Proposition A3.3 (with $n \triangleq lk$, $m \triangleq kd$, $\beta \triangleq \alpha(q)$, and $\hat{M}=c_{10}k^{3/2}$), there exists $c_{11}>0$, depending only on constants M, N, D, d, and B_1', B_2' of Section 2 such that for every $0<\zeta<1$,

$$\begin{aligned}
&\Pi^{kd}_{\infty}\left(\mathscr{L}\left(\varepsilon^{1/2}\sum_{j=1}^{k}\sum_{i=1}^{l}\tilde{Y}^{\varepsilon,k}_{i,j}\right), \mathscr{N}\left(0, \varepsilon\sum_{j=1}^{k}\sum_{i=1}^{l}\operatorname{cov}(\tilde{Y}^{\varepsilon,k}_{i,j})\right)\right)\\
&\quad\leqslant \Pi^{kd}_{2}\left(\mathscr{L}\left((kl)^{-1/2}\sum_{j=1}^{k}\sum_{i=1}^{l}\tilde{X}^{\varepsilon,k}_{i,j}\right), \mathscr{N}\left(0, (kl)^{-1}\sum_{j=1}^{k}\sum_{i=1}^{l}\operatorname{cov}(\tilde{X}^{\varepsilon,k}_{i,j})\right)\right)\\
&\quad\leqslant c_{11}\Big\{\alpha^{2/3}(q)(kl)^{1/2}\zeta^{-1}(kd)^{-1/2}k^{1/2}\\
&\qquad+(kd)^{3/2}\alpha^{1/3}(q)k\zeta^{-2}+(kl)^{-1/2}k^{3/2}\zeta^{-3}(kd)^{-1/2}\\
&\qquad+4\zeta+4\zeta\left(\log\frac{1}{\zeta}\right)^{1/2}(kd)^{1/2}+4\zeta(kd)^{1/2}\Big\}.
\end{aligned} \tag{A6.47}$$

Now $q=k^{-1}\varepsilon^{-1/4}\geqslant 1$ by the choice of k and ε, so by condition (C4) of Section 2,

$$\alpha(q)\leqslant \eta q^{-2}=\eta k^2\varepsilon^{1/2}, \tag{A6.48}$$

where $\eta>0$ is the constant of condition (C4). Substituting (A6.48), (A6.22), and (A6.24) into (A6.47) and taking $\zeta \triangleq \varepsilon^{1/32}$,

$$\begin{aligned}
&\Pi^{kd}_{\infty}\left(\mathscr{L}\left(\varepsilon^{1/2}\sum_{j=1}^{k}\sum_{i=1}^{l}\tilde{Y}^{\varepsilon,k}_{i,j}\right), \mathscr{N}\left(0, \varepsilon\sum_{j=1}^{k}\sum_{i=1}^{l}\operatorname{cov}(\tilde{Y}^{\varepsilon,k}_{i,j})\right)\right)\\
&\quad\leqslant c_{11}\Big\{\eta^{2/3}k^{11/6}\varepsilon^{17/96}d^{-1/2}+\eta^{1/3}k^{19/6}\varepsilon^{5/48}d^{3/2}+2k^{1/2}\varepsilon^{1/32}d^{-1/2}\\
&\qquad+4\varepsilon^{1/32}+\frac{\varepsilon^{1/32}}{2^{1/2}}\left[\log\frac{1}{\varepsilon}\right]^{1/2}(kd)^{1/2}+4k^{1/2}\varepsilon^{1/32}d^{1/2}\Big\}.
\end{aligned} \tag{A6.49}$$

Now, concentrating on the fifth term on the right of (A6.49), we use the fact that for any $\kappa>0$ there exists some $x_0(\kappa)$ such that

$$x^{-\kappa}(\log x)^{1/2}\leqslant 1 \qquad \text{for all} \quad x\geqslant x_0(\kappa). \tag{A6.50}$$

Letting $x=\varepsilon^{-1}$, $\varepsilon_1(\kappa)=1/x_0(\kappa)$, and $\kappa=\frac{1}{32}-\frac{1}{33}$ in (A6.50), we obtain

$$\varepsilon^{1/32}[\log\varepsilon^{-1}]^{1/2}\leqslant\varepsilon^{1/33}\qquad\text{for all}\quad 0<\varepsilon\leqslant\varepsilon_1. \tag{A6.51}$$

Thus, if we choose $K_1=\max\{\varepsilon_1^{-1/4},2\}$, from (A6.49) and (A6.51) there exists $c_{12}>0$ depending only on M, N, D, d, η, and B_1', B_2' of Section 2 such that

$$\Pi_\infty^{kd}\left(\mathscr{L}\left(\varepsilon^{1/2}\sum_{j=1}^{k}\sum_{i=1}^{l}\tilde{Y}_{i,j}^{\varepsilon,k}\right),\mathscr{N}\left(0,\varepsilon\sum_{j=1}^{k}\sum_{i=1}^{l}\operatorname{cov}(\tilde{Y}_{i,j}^{\varepsilon,k})\right)\right)\leqslant c_{12}k^{19/6}\varepsilon^{1/33} \tag{A6.52}$$

for all $0<\varepsilon\leqslant k^{-4}$, $k=K_1, K_1+1, \ldots$.

Now consider the third term on the right-hand side of (A6.31) and fix a positive integer $k\geqslant 1$ and a $0<\varepsilon\leqslant\frac{1}{16}$. Define

$$\tilde{T}^{\varepsilon,k}\triangleq\varepsilon\sum_{j=1}^{k}\sum_{i=1}^{l}\operatorname{cov}(\tilde{Y}_{i,j}^{\varepsilon,k}) \tag{A6.53}$$

and

$$\tilde{S}^k\triangleq\operatorname{cov}(\hat{\Xi}_k^0)=E\{(\hat{\Xi}_k^0)(\hat{\Xi}_k^0)^T\}. \tag{A6.54}$$

Then by the inequality in Remark A3.1,

$$\begin{aligned}|||\tilde{T}^{\varepsilon,k}-\tilde{S}^k|||_1&\leqslant kd\sqrt{\sum_{n,m=1}^{k}|T_{n,m}^{\varepsilon,k}-S_{n,m}^k|_2^2}\\&\leqslant k^2 d\max_{1\leqslant n\leqslant m\leqslant k}|T_{n,m}^{\varepsilon,k}-S_{n,m}^k|_2,\end{aligned} \tag{A6.55}$$

where $T_{n,m}^{\varepsilon,k}$ and $S_{n,m}^k$ represent the d by d submatrices of elements $\{(n-1)d+1,\ldots,nd\}\times\{(m-1)d+1,\ldots,md\}$ of $\tilde{T}^{\varepsilon,k}$ and $\tilde{S}^k$, respectively. Fix $1\leqslant n\leqslant m\leqslant k$. By (2.14) $\{\hat{W}^0(\tau),\ 0\leqslant\tau\leqslant 1\}$ is a zero mean Gaussian process with independent increments and covariance $\int_0^\tau A(x^0(s))\,ds$ such that $\hat{W}^0(0)=0$, so from (A6.54), (A6.21), (A6.25), (A6.26), and (A6.27),

$$\begin{aligned}S_{n,m}^k&=E(\hat{W}^0(n/k)-\hat{W}^0(0))(\{\hat{W}^0(n/k)-\hat{W}^0(0)\}\\&\qquad+\{\hat{W}^0(m/k)-\hat{W}^0(nk)\})^T\\&=E\{\hat{W}^0(n/k)(\hat{W}^0(n/k))^T\}=\int_0^{n/k}A(x^0(s))\,ds\\&=\varepsilon\left[\sum_{j=1}^{n}\left(\sum_{i=1}^{l}\int_{H_{i,j}^{\varepsilon,k}}A(x^0(\varepsilon u))\,du+\sum_{i=1}^{l+1}\int_{I_{i,j}^{\varepsilon,k}}A(x^0(\varepsilon u))\,du\right)\right].\end{aligned} \tag{A6.56}$$

Now by Lemma A7.1 there exists constant B such that $|A(x)|_2 \leqslant B$ for $|x| \leqslant D$ and so by (A6.56), (A6.26), (A6.27), (A6.24), and (A6.22),

$$\left| S^k_{n,m} - \varepsilon \sum_{j=1}^{n} \sum_{i=1}^{l} \int_{H^{\varepsilon,k}_{i,j}} A(x^0(\varepsilon s))\, ds \right|_2 \leqslant \varepsilon \sum_{j=1}^{n} \sum_{i=1}^{l+1} \int_{I^{\varepsilon,k}_{i,j}} |A(x^0(\varepsilon s))|_2, ds$$

$$\leqslant B\varepsilon \sum_{j=1}^{n} (lq+p+q) \leqslant B \frac{n}{k} (\varepsilon^{1/2} + \varepsilon^{1/4} + \varepsilon^{3/4}) \leqslant 3B\varepsilon^{1/4}. \qquad \text{(A6.57)}$$

Now by (A6.30) and the fact that $1 \leqslant n \leqslant m \leqslant k$, for any $i = 1, ..., l$ and $j = 1, ..., k$, we obtain

$$[\operatorname{cov}(\tilde{Y}^{\varepsilon,k}_{i,j})]_{n,m} = \begin{cases} 0 & \text{if } n < j, \\ \operatorname{cov}(Y^{\varepsilon,k}_{i,j}) & \text{if } n \geqslant j, \end{cases} \qquad \text{(A6.58)}$$

where $[\operatorname{cov}(\tilde{Y}^{\varepsilon,k}_{i,j})]_{n,m}$ denotes the d by d matrix of elements $\{(n-1)d+1, nd\} \times \{(m-1)d+1, md\}$ of $\operatorname{cov}(\tilde{Y}^{\varepsilon,k}_{i,j})$. Thus, letting $\tau_{ij} \triangleq (j-1)\varepsilon^{-1}k^{-1} + (i-1)(p+q)$ we have by (A6.53), (A6.58), (A6.28), (A6.25), Lemma A6.3 (to follow), and (A6.24) that there exist constants $c_{13} > 0$ and $\beta \triangleq \chi/3$ (since $\tau_{ij} + k^{-1}\varepsilon^{-3/4} < \varepsilon^{-1}$) such that

$$\left| T^{\varepsilon,k}_{n,m} - \varepsilon \sum_{j=1}^{n} \sum_{i=1}^{l} \int_{H^{\varepsilon,k}_{i,j}} A(x^0(\varepsilon s))\, ds \right|_2$$

$$= \varepsilon \left| \sum_{j=1}^{k} \sum_{i=1}^{l} [\operatorname{cov}(\tilde{Y}^{\varepsilon,k}_{i,j})]_{n,m} - \sum_{j=1}^{n} \sum_{i=1}^{l} \int_{H^{\varepsilon,k}_{i,j}} A(x^0(\varepsilon s))\, ds \right|_2$$

$$\leqslant \varepsilon \sum_{j=1}^{n} \sum_{i=1}^{l} \left| E\left(\int_{\tau_{ij}}^{\tau_{ij}+k^{-1}\varepsilon^{-3/4}} \tilde{F}(x^0(\varepsilon s), s)\, ds \right) \left(\int_{\tau_{ij}}^{\tau_{ij}+k^{-1}\varepsilon^{-3/4}} \tilde{F}(x^0(\varepsilon t), t)\, dt \right)^T \right.$$

$$\left. - \int_{\tau_{ij}}^{\tau_{ij}+k^{-1}\varepsilon^{-3/4}} A(x^0(\varepsilon s))\, ds \right|_2$$

$$\leqslant \varepsilon \sum_{j=1}^{n} \sum_{i=1}^{l} c_{13} \varepsilon^{-3(1-\beta)/4} \leqslant c_{13} k \varepsilon^{3\beta/4}. \qquad \text{(A6.59)}$$

Therefore, since (A6.57) and (A6.59) hold for all $1 \leqslant n \leqslant m \leqslant k$ we obtain from (A6.55) a constant $c_{14} > 0$ such that

$$|||\tilde{T}^{\varepsilon,k} - \tilde{S}^k|||_1 \leqslant k^2 \{3B\varepsilon^{1/4} + c_{13} k \varepsilon^{3\beta/4}\} \leqslant c_{14} k^3 \varepsilon^{\sigma}, \qquad \text{(A6.60)}$$

where $\sigma \triangleq \min\{1/4, 3\beta/4\}$. Now, by (A6.50), there exists some $x_0 > 1$ such that

$$x^{-1/3} (\log x)^{1/2} \leqslant x^{-1/4} \qquad \text{for all} \quad x \geqslant x_0. \qquad \text{(A6.61)}$$

Moreover, by (A6.60) there exists positive integer K_2 such that

$$\inf_{0<\varepsilon\leqslant k^{-19/3\sigma}} \frac{k}{|||\tilde{T}^{\varepsilon,k}-\tilde{S}^k|||_1} \geqslant \frac{1}{c_{14}} k^{19/3-2} \geqslant x_0 \tag{A6.62}$$

for all $k \geqslant K_2$. Thus by (A6.53), (A6.54), Theorem A3.4, (A6.62), (A6.61), and (A6.60), there exist constants c_{15} and c_{16} depending only on η and θ of condition (C4) in Section 2 and c_{14} such that

$$\begin{aligned} &\Pi_\infty^{kd}\left(\mathcal{N}\left(0, \varepsilon \sum_{j=1}^{k} \sum_{i=1}^{l} \operatorname{cov}(\tilde{Y}_{i,j}^{\varepsilon,k})\right), \mathcal{L}(\hat{\Xi}_k^0)\right) \\ &\leqslant \Pi_2^{kd}\left(\mathcal{N}\left(0, \varepsilon \sum_{j=1}^{k} \sum_{i=1}^{l} \operatorname{cov}(\tilde{Y}_{i,j}^{\varepsilon,k})\right), \mathcal{L}(\hat{\Xi}_k^0)\right). \\ &\leqslant c_{15}\, |||\tilde{T}^{\varepsilon,k}-\tilde{S}^k|||_1^{1/3}\, k^{1/6}\left\{1+\left|\log\left(\frac{k}{|||\tilde{T}^{\varepsilon,k}-S^k|||_1}\right)\right|^{1/2}\right\} \\ &\leqslant c_{15}\, |||\tilde{T}^{\varepsilon,k}-\tilde{S}^k|||_1^{1/3}\, k^{1/6} + c_{15} k^{1/2} k^{-1/4} \{|||\tilde{T}^{\varepsilon,k}-\tilde{S}^k|||_1\}^{1/4} \\ &\leqslant c_{16} k^{7/6} \varepsilon^{\sigma/4} \qquad \text{for all} \quad 0<\varepsilon\leqslant k^{-19/3\sigma}, \quad k \geqslant K_2. \end{aligned} \tag{A6.63}$$

The lemma follows from (A6.31), (A6.40), (A6.52), and (A6.63) by taking $K_0 = \max\{K_1, K_2\}$ and $\mu = \min(1/33, \sigma/4\} = \min\{1/33, 3\beta/16\}$, noting that $\beta = \chi/3$ in (A6.59) and the fact that $k^{-19/3\mu} \leqslant \min\{k^{-4}, k^{-19/3\sigma}\}$ for $k = K_0$, K_0+1, K_0+2, ... ∎

The following lemma is a technical result used in line (A6.59) of Lemma A6.2.

LEMMA A6.3. *Under conditions* (C0)–(C6) *of Section* 2, *there exist constants* $c_{17} > 0$ *and* $\beta \triangleq \chi/3$ *such that for any integers* $1 \leqslant n, m \leqslant d$,

$$\left| E\left(\int_{t_0}^{t_0+\tau\varepsilon^{-3/4}} \int_{t_0}^{t_0+\tau\varepsilon^{-3/4}} \tilde{F}_m(x^0(\varepsilon s), s)\tilde{F}_n(x^0(\varepsilon t), t)\, dt\, ds\right) - \int_{t_0}^{t_0+\tau\varepsilon^{-3/4}} A_{m,n}(x^0(\varepsilon s))\, ds \right| \leqslant c_{17}\varepsilon^{-3(1-\beta)/4}$$

for all $t_0 \geqslant 0$, $0 \leqslant \tau \leqslant 1$, *and* $0 < \varepsilon \leqslant 1$ *such that* $\varepsilon(t_0 + \tau\varepsilon^{-3/4}) \leqslant 1$. *Here* $A(\cdot)$ *and* χ *are defined by condition* (C6) *of Section* 2.

Proof. Fix integers m, n such that $1 \leqslant m$, $n \leqslant d$ and fix $t_0 \geqslant 0$, $0 \leqslant \tau \leqslant 1$, and $0 < \varepsilon \leqslant 1$ such that $\varepsilon(t_0 + \tau\varepsilon^{-3/4}) \leqslant 1$. The following definitions are made without emphasizing the dependence on ε and τ when convenient:

$$\eta_\varepsilon \triangleq [\varepsilon^{-3/8}], \qquad \Delta(\varepsilon, \tau) \triangleq \frac{\tau\varepsilon^{-3/4}}{\eta_\varepsilon} \leqslant 2\tau\varepsilon^{-3/8}, \tag{A6.64}$$

$$\sigma_0 \triangleq t_0, \qquad \sigma_{i+1} \triangleq \sigma_i + \Delta(\varepsilon, \tau) \qquad \text{for} \quad i = 0, 1, ..., \eta_\varepsilon - 1, \tag{A6.65}$$

$$A \triangleq \bigcup_{i=0}^{n_\varepsilon - 1} A_i, \qquad \text{where} \quad A_i \triangleq (\sigma_i, \sigma_{i+1}] \times (\sigma_i, \sigma_{i+1}]$$
$$\text{for} \quad i = 0, 1, ..., \eta_\varepsilon - 1, \tag{A6.66}$$

$$B \triangleq ([t_0, t_0 + \tau\varepsilon^{-3/4}] \times [t_0, t_0 + \tau\varepsilon^{-3/4}]) \sim A. \tag{A6.67}$$

Also, for ease of notation, define (without emphasizing the dependence on (m, n)):

$$\Psi_\varepsilon(s, t) \triangleq E\{\tilde{F}_m(x^0(\varepsilon s), s)\, \tilde{F}_n(x^0(\varepsilon t), t)\} \qquad \text{for} \quad 0 \leqslant s, t \leqslant \varepsilon^{-1}. \tag{A6.68}$$

Then, by (A6.68), (A6.67), Fubini's theorem, and (A6.66),

$$\left| E\left(\int_{t_0}^{t_0 + \tau\varepsilon^{-3/4}} \int_{t_0}^{t_0 + \tau\varepsilon^{-3/4}} \tilde{F}_m(x^0(\varepsilon s), s) \tilde{F}_n(x^0(\varepsilon t), t)\, dt\, ds\right) - \int_A\int \Psi_\varepsilon(s, t)\, dt\, ds \right|$$
$$= \left| \int_B\int \Psi_\varepsilon(s, t)\, dt\, ds \right| = \left| 2 \sum_{i=1}^{n_\varepsilon - 1} \int_{\sigma_0}^{\sigma_i} \int_{\sigma_i}^{\sigma_{i+1}} \Psi_\varepsilon(s, t)\, dt\, ds \right|. \tag{A6.69}$$

However, since $E\tilde{F}_m(x^0(\varepsilon s), s) = E\tilde{F}(x^0(\varepsilon t), t) = 0$ we obtain from Lemma A3.2 of Appendix 3 and Lemma A7.2(ii) of Appendix 7,

$$|\Psi_\varepsilon(s, t)| \leqslant 10\{\alpha(|s-t|)\}^{\delta/(\delta+2)} \cdot \|\tilde{F}_m(x^0(\varepsilon s), s)\|_{2+\delta} \cdot \|\tilde{F}_n(x^0(\varepsilon t), t)\|_{2+\delta}$$
$$\leqslant 10(2M + 2ND)^2 \{\alpha(|s-t|)\}^{\delta/(\delta+2)} \tag{A6.70}$$

for all $t_0 \leqslant s$, $t \leqslant t_0 + \tau\varepsilon^{-3/4}$, where $\delta > 0$ is the constant of condition (C3) in Section 2. By (A6.69), (A6.70), and (2.5) there exists $c_{18} > 0$ such that

$$\left| E\left(\int_{t_0}^{t_0 + \tau\varepsilon^{-3/4}} \int_{t_0}^{t_0 + \tau\varepsilon^{-3/4}} \tilde{F}_m(x^0(\varepsilon s), s)\, \tilde{F}_n(x^0(\varepsilon t), t)\, dt\, ds\right)^2 \right.$$
$$\left. - \int_A\int \Psi_\varepsilon(s, t)\, dt\, ds \right|$$
$$\leqslant c_{18} \sum_{i=1}^{\eta_\varepsilon - 1} \int_{\sigma_0}^{\sigma_i} \int_{\sigma_i}^{\sigma_{i+1}} [\alpha(s-t)]^{\delta/(\delta+2)}\, ds\, dt$$
$$\leqslant c_{18} \sum_{i=1}^{\eta_\varepsilon - 1} \int_{-\infty}^{\sigma_i} \int_{\sigma_i}^{\infty} [\alpha(s-t)]^{\delta/(\delta+2)}\, ds\, dt$$
$$= c_{18}\eta_\varepsilon \int_0^\infty \tau[\alpha(\tau)]^{\delta/(\delta+2)}\, d\tau \leqslant c_{18}\varepsilon^{-3/8} B_2'. \tag{A6.71}$$

Moreover, by (2.10), the fact that $x \to \tilde{F}_m(x, s)$ has a global Lipschitz bound of $2N$, (A6.64), and Lemma A7.2(ii), (iii),

$$\begin{aligned}
&\left| \int_A \int \Psi_\varepsilon(s, t)\, dt\, ds - \sum_{i=0}^{\eta_\varepsilon - 1} \Delta(\varepsilon, \tau) A_{m,n}(x^0(\varepsilon\sigma_i)) \right| \\
&\quad \leqslant \sum_{i=0}^{\eta_\varepsilon - 1} \left| \int_{\sigma_i}^{\sigma_{i+1}} \int_{\sigma_i}^{\sigma_{i+1}} E\{\tilde{F}_m(x^0(\varepsilon\sigma_i), s) \tilde{F}_n(x^0(\varepsilon\sigma_i), t)\}\, dt\, ds \right. \\
&\qquad \left. - \Delta(\varepsilon, \tau) A_{m,n}(x^0(\varepsilon\sigma_i)) \right| \\
&\qquad + \sum_{i=0}^{\eta_\varepsilon - 1} \int_{\sigma_i}^{\sigma_{i+1}} \int_{\sigma_i}^{\sigma_{i+1}} |E\{\tilde{F}_m(x^0(\varepsilon s), s) \tilde{F}_n(x^0(\varepsilon t), t)\} \\
&\qquad - E\{\tilde{F}_m(x^0(\varepsilon\sigma_i), s)\, \tilde{F}_n(x^0(\varepsilon\sigma_i), t)\}|\, dt\, ds \\
&\quad \leqslant \gamma \eta_\varepsilon\, \Delta(\varepsilon, \tau)^{1-\chi} + \sum_{i=0}^{\eta_\varepsilon - 1} \int_{\sigma_i}^{\sigma_{i+1}} \int_{\sigma_i}^{\sigma_{i+1}} \\
&\qquad \{E\, |\tilde{F}_m(x^0(\varepsilon s), s)| \cdot 2N\, |x^0(\varepsilon t) - x^0(\varepsilon\sigma_i)| \\
&\qquad + 2N\, |x^0(\varepsilon s) - x^0(\varepsilon\sigma_i)| \cdot E\, |\tilde{F}_n(x^0(\varepsilon\sigma_i), t)|\}\, dt\, ds \\
&\quad \leqslant \gamma \varepsilon^{-3/8} 2\varepsilon^{-(3/8)(1-\chi)} + 8\varepsilon\eta_\varepsilon\, \Delta^3(\varepsilon, \tau) N(M + ND)^2 \\
&\quad \leqslant c_{19}(\varepsilon^{-(3/4)(1-(1/2)\chi)} + \varepsilon^{-1/2})
\end{aligned} \tag{A6.72}$$

for some constant $c_{19} > 0$ depending only on N, M, D, d, and γ of Section 2. Moreover, by Lemma A7.1 (with $R \triangleq D$), there exists constant $c_{20} > 0$ such that

$$|A_{m,n}(x') - A_{m,n}(x)| \leqslant c_{20}\, |x' - x| \qquad \text{for all} \quad |x'|, |x| \leqslant D \tag{A6.73}$$

and so by (A6.73), (A6.65), Lemma A7.2(iii), and (A6.64),

$$\begin{aligned}
&\left| \int_{t_0}^{t_0 + \tau\varepsilon^{-3/4}} A_{m,n}(x^0(\varepsilon s))\, ds - \sum_{i=0}^{\eta_\varepsilon - 1} \Delta(\varepsilon, \tau) A_{m,n}(x^0(\varepsilon\sigma_i)) \right| \\
&\quad \leqslant \sum_{i=0}^{\eta_\varepsilon - 1} \int_{\sigma_i}^{\sigma_{i+1}} |A_{m,n}(x^0(\varepsilon s) - A_{m,n}(x^0(\varepsilon\sigma_i))|\, ds \\
&\quad \leqslant \eta_\varepsilon \int_0^{\Delta(\varepsilon,\tau)} c_{20}(M + ND)\, \varepsilon s\, ds \\
&\quad = \frac{c_{20}}{2} \eta_\varepsilon\, \Delta^2(\varepsilon, \tau)(M + ND)\varepsilon \leqslant c_{20}(M + ND)\varepsilon^{-1/8}.
\end{aligned} \tag{A6.74}$$

Finally, by (A6.71), (A6.72), and (A6.74) there exists constant $c_{21}>0$ such that

$$\left| E\int_{t_0}^{t_0+\tau\varepsilon^{-3/4}} \int_{t_0}^{t_0+\tau\varepsilon^{-3/4}} \tilde{F}_m(x^0(\varepsilon s), s)\tilde{F}_n(x^0(\varepsilon t), t)\, dt\, ds - \int_{t_0}^{t_0+\tau\varepsilon^{-3/4}} A_{m,n}(x^0(\varepsilon s))\, ds \right| \leqslant c_{21}(\varepsilon^{-1/2}+\varepsilon^{-(3/4)(1-\chi/2)}) \quad \text{(A6.75)}$$

and the lemma follows with $\beta \triangleq \chi/3$, since $\max\{\frac{1}{2}, \frac{3}{4}(1-\chi/2)\} \leqslant \frac{3}{4}(1-\chi/3)$ for $0 \leqslant \chi \leqslant 1$. ∎

APPENDIX 7: MISCELLANEOUS TECHNICAL RESULTS

This appendix contains two useful technical results. The first lemma establishes a Lipschitz bound, on any compact domain of $\Re^d$, for the function $A(\cdot)$ defined in Eq. (2.10). It is used to obtain lines (3.51) and (A6.73).

LEMMA A7.1. *Under the conditions* (C0)–(C6) *of Section* 2, *for any fixed* $R>0$ *there exists constant* $c_1>0$ *depending only on* R *and the constants* M, N, d, *and* B_1', B_1 *of Section* 2 *such that*

$$\max_{1\leqslant i,j\leqslant d} |A_{i,j}(x)-A_{i,j}(x')| \leqslant c_1\, |x-x'| \quad \textit{for all } x, x', \textit{ where } |x|, |x'| \leqslant R.$$

Proof. Fix two points x and x' as in the problem statement and integers i, j such that $1 \leqslant i,\ j \leqslant d$. By (2.10) and Fubini,

$$\begin{aligned} &|A_{i,j}(x)-A_{i,j}(x')| \\ &\quad = \left| \lim_{T\to\infty} \frac{1}{T}\int_0^T\int_0^T E\tilde{F}_i(x,s)\tilde{F}_j(x,t) - E\tilde{F}_i(x',s)\tilde{F}_j(x',t)\, ds\, dt \right| \\ &\quad \leqslant \limsup_{T\to\infty} \frac{1}{T}\left| E\left\{ \int_0^T\int_0^T \tilde{F}_i(x,s)(\tilde{F}_j(x,t)-\tilde{F}_j(x',t))\, ds\, dt \right\}\right| \\ &\quad\quad + \limsup_{T\to\infty} \frac{1}{T}\left| E\left\{ \int_0^T\int_0^T (\tilde{F}_i(x,s)-\tilde{F}_i(x',s))\tilde{F}_j(x',t)\, ds\, dt \right\}\right|. \end{aligned} \quad \text{(A7.1)}$$

Now for all $0 \leqslant s \leqslant T$, $|\tilde{F}_j(x,s)-\tilde{F}_j(x',s)| \leqslant 2N\,|x-x'|$ and $\|\tilde{F}_i(x,s)\|_{2+\delta} \leqslant 2M+2NR$ by Lemma A7.2(i), so by the Cauchy–Schwarz inequality, Fubini's theorem, and Lemma A4(A) and (B), there exists constant $c_2 > 0$ depending only on R, M, N, d, and B_1', B_1 such that

$$\begin{aligned}
&\left| E\left\{ \int_0^{CT} \tilde{F}_i(x, s)\, ds \int_0^T (\tilde{F}_j(x, t) - \tilde{F}_j(x', t))\, dt \right\} \right| \\
&\quad \leqslant \left(\int_0^T \int_0^T E\tilde{F}_i(x, s)\tilde{F}_i(x, t)\, ds\, dt \right. \\
&\qquad \left. \times \int_0^T \int_0^T E(\tilde{F}_j(x, s) - \tilde{F}_j(x', s))(\tilde{F}_j(x, t) - \tilde{F}_j(x', t))\, ds\, dt \right)^{1/2} \\
&\quad \leqslant c_2 \sqrt{T \cdot |x - x'|^2\, T} = c_2 T\, |x - x'|. \qquad \text{(A7.2)}
\end{aligned}$$

The lemma follows from (A7.1) and (A7.2). ▌

The following technical lemma extends the uniform moment bound on $\{\tilde{F}_i(0, t),\ t \geqslant 0\}$ (given in condition (C3) of Section 2) to a uniform moment bound on $\{\tilde{F}_i(x^0(s\varepsilon), s),\ 0 \leqslant s \leqslant \varepsilon^{-1}\}$ using the Lipschitz condition (2.1) and (for (i), (ii)) the Minikowski inequality.

LEMMA A7.2. *Assume conditions* (C0), (C1), (C3), *and* (C5) *in Section* 2 *and suppose that* $1 \leqslant \lambda \leqslant (8 + 4\delta)$. *Then*

(i) $\sup_{1 \leqslant i \leqslant d} \sup_{t \leqslant 0} \|\tilde{F}_i(x, t)\|_\lambda \leqslant 2M + 2N\,|x|$ for all $x \in \mathfrak{R}^d$,

(ii) $\sup_{1 \leqslant i \leqslant d} \sup_{T \geqslant 0} \sup_{0 \leqslant t \leqslant T} \|F_i(x^0(t/T), t)\|_\lambda \leqslant 2M + 2ND$,

(iii) $|x^0(\tau) - x^0(\tau')| \leqslant (M + ND)\,|\tau - \tau'|$ *for all* $0 \leqslant \tau,\ \tau' \leqslant 1$,

where $\tilde{F}(x, t)$ *is given by* (2.9) *and the constants* δ, M, N, d, *and* D *are from Section* 2.

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